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THE SIMILARITY SOLUTION FOR A CONVERGENT SPHERICAL SHOCK WAVE NEAR ZERO RADIUS

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ABSTRACT

The problem of a convergent spherical shock wave, near zero radius, is studied by means of the similarity method. It is found that if a fixed (p, v)-relation (i.e., independent of entropy) is assumed then the method necessitates taking both pressure and density to be zero ahead of the shock. Obviously, then, the supposed shock becomes a free surface and the problem reduces to that solved in LA-210. In the present report, therefore, p, v, are taken to be related by an equation

$$p = kv^{-\gamma} \tag{1a}$$

where k is a function of entropy. The nature of this function need not be specified, since the assumption of similarity determines the dependence of k on the mass of unshocked material (i.e., on the variable x introduced below).

The assumption of similarity may be described as follows: With f defined by

$$y = aft^n, \qquad a = \text{const.} \tag{2a}$$

where y is radius, the position of the convergent shock is taken to be given by

$$f = f_o = \text{const} \tag{3a}$$

the sense of time being reversed, and the variables v specific volume, p pressure, u material velocity, are assumed to have the form

$$v = v_1(f)t^{q_1}, \quad p = p_1(f)t^{q_2}, \quad u = u_1(f)t^{q_3} \tag{4a}$$

One finds readily $q_1 = 0$, $q_2 = 2(n - 1)$, $q_3 = n - 1$. Methods quite analogous to those of LA-210 serve in this instance also to reduce the hydrodynamical equations to a single first-order ordinary differential equation containing the parameter n. Since $v = v_1(f_o)$ is constant on the shock $f = f_o$, its value μ is also a parameter, and the general problem is to determine the pairs μ, n,

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for which physically satisfactory solutions exist. This problem is discussed in complete detail analytically, and numerical results obtained for various γ. The phenomenon of a continuous spectrum of values of n first observed in LA-210 makes its appearance here also, but as in that report, the imposition of the condition of analyticity appears to specify a unique n in any given problem. With regard to the value of μ, for a perfect gas, $\gamma \leqq 5/3$, of course $\mu = (\gamma - 1)/(\gamma + 1)$, but for other applications envisaged, this condition need not be enforced. It is found, however, that always $\mu < \gamma/(\gamma + 1)$, since otherwise the shock is not subsonic with reference to the material behind it. Moreover, if $p = kv^{-\gamma}$ is taken to be universal, rather than valid only for $v \ll \infty$, then $\mu \leqq (\gamma - 1)/(\gamma + 1)$.

The analytic solution for $\gamma = 3$, $\mu = .497$, when $n = .636$, is determined completely and the results presented in the form of a table and graphs. In order that these, as well as the graphs dealing with the (μ, n)-relation can be understood without a detailed reading of the report, a short explanation is given separately (pp 47-8).

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1. Description of the Problem. Suppose a strong shock of uniform intensity to be communicated to the surface of a sphere of inert material. A shock wave will then be induced in the sphere, travel inwards with spherical symmetry about the center, and ultimately be reflected from that point. Given the initial strength of the shock and the requisite thermodynamic information about the substance composing the sphere, the hydrodynamical problem of the motion which takes place is completely defined, and susceptible of numerical integration over a considerable portion of the inward motion[1]. The center of the sphere, however, is a singularity of the hydrodynamical equations, and in consequence straightforward numerical integration is not indefinitely possible. One has, therefore, to resort to other means to determine the motion near this point, and it is this problem with which we are concerned.

To investigate it we employ the so-called "similarity" method[2]. Taking the point to which the shock converges as origin, and reversing the sense of time, t, the assumption of similarity amounts to this: that the equation of the incoming shock has the form

$$y = a\ f_o\ t^n \qquad (y = \text{radius}) \qquad (1.1)$$

and that, with f defined by

$$y = a\ f\ t^n \qquad (1.2)$$

1) Cf. a forthcoming report by Christy.
2) Cf., for example, LA-210, "The Similarity Solution for a Convergent Free Surface near Zero Radius", and BM 210, "Powerful Spherical and Cylindrical Shocks in the Neighborhood of the Center of the Sphere and of the Cylinder Axis", G. Guderley. Translated by M. Flint from Luftfahrforschung, vol. 19, No. 9, 20. 10. 42, pp. 301 - 312.

behind the shock, the variables p(pressure), v (specific volume), and u (velocity) are of the form

$$p = p'(f)t^{q_1}, \quad v = v'(f)t^{q_2}, \quad u = u'(f)t^{q_3} \qquad (1.3)$$

Here a is a scaling factor which it is convenient to leave free. Continuing the assumption of similarity beyond $t = 0$, we also expect the reflected shock to have the form (1.1), and the form of the functions (1.3) to remain unchanged, time being measured now in the ordinary sense. The problem then is to determine the exponents n, q_1, q_2, q_3, and the functions p', v', u'.

2. Shock Wave Boundary Conditions and Their Implications.

Across any shock wave, the quantities p, u, v, are discontinuous, with the discontinuities subject to the conservation laws of Rankine, Hugoniot. Let the values of a variable on the two sides of the shock be distinguished by the subscripts 1, 2; let U denote shock velocity, and E the internal energy of the substance. Then these laws can be written as follows:

(Conservation of Mass) $$(v_1 - v_2)U = v_1u_2 - v_2u_1 \qquad (2.1)$$

(Conservation of Momentum) $$(v_1 - v_2)(p_2 - p_1) = \rho_o(u_2 - u_1)^2 \qquad (2.2)$$

(Conservation of Energy) $$E_2 - E_1 = (1/2)(p_1 + p_2)(v_1 - v_2) \qquad (2.3)$$

In (2.2), which is not the direct expression of the conservation of momentum, since it takes account of (2.1), ρ_o denotes normal density.

Our present concern is only with (2.1), (2.2). First, we observe that for the incoming shock, we have $u_1 = 0$, $v_1 = v_o$ = constant, so that (2.1) becomes simply (2.4)

$$(1 - v_2 v_o^{-1})U = u_2 \qquad (2.4)$$

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Hence it is at once clear that a solution of the form specified in (1.1), (1.2), (1.3) is possible only if

$$v'(f) = 0 \quad \text{or } q_2 = 0; \quad q_3 = n - 1 \tag{2.5}$$

But if $v'(f) = 0$, then (2.4) becomes $U = u_2$, which states that no material crosses the shock. Furthermore, taking $v'(f_o) = 0$ means in effect that we are neglecting entirely the density of the material ahead of the shock. Hence, under the only reasonable interpretation, (2.2) reduces to $p_2 = 0$.

Now, the conditions $U = u_2$, $p_2 = 0$ are, in fact, the conditions for a free surface and thus the alternative $v'(f) = 0$ leads to the motion of such a surface as the asymptotic limiting form of a convergent spherical shock. This is not entirely unreasonable, moreover; for while the condition $p_2 = 0$ seems inconsistent with the concept of a shock, it is to be remembered that behind such a surface, the pressure rises very steeply to a maximum which becomes infinite as $t \to 0$, and of course the distance of the maximum from the free surface approaches zero. Thus it is only very near the surface that the behavior of the pressure is qualitatively different from that behind a shock. The problem to which this alternative leads, however, has already been solved (Cf. LA-210), and we shall not, therefore, be further concerned with it here.

Rather, we devote our entire attention to the case $q_2 = 0$, where v is a constant on each similarity curve, including, in particular, the shock. Taking $v_1 = 1$, $v_2 = \mu$, $p_1 = 0$ the equation (2.2) for the incoming shock becomes in this case

$$(1 - \mu)p_2 = \rho_o u_2^2, \tag{2.6}$$

and in consequence $q_1 = 2(n - 1)$. Thus q_1, q_2, q_3 are all determined in terms

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of n: summing up we have

$$q_1 = 2(n - 1), \qquad q_2 = 0, \qquad q_3 = n - 1 \tag{2.7}$$

In view of the singularity at the origin, moreover, we should have $0 < n < 1$, and hence, since p depends on t, while v does not, a fixed (p,v)-equation behind the shock is impossible. In other words, under the assumption of similarity, and the exclusion of the free-surface solution already discussed, one cannot neglect the dependence of the pressure on entropy. Accordingly, in treating the alternative $q_2 = 0$, we are led to assume an adiabatic equation of state of the form

$$p = kv^{-\gamma} \tag{2.8}$$

where k is a function of entropy, S.

The applications of the present analysis are, therefore, limited to those cases in which (2.8) is at least a fair approximation. This includes in particular, of course, for $1 < \gamma \leq 5/3$, the case of a perfect gas[3]; in addition it is hoped that with $\gamma \sim 3$, (2.8) may be approximately true for various metals. Finally, a convergent detonation wave may be regarded in the limit as a pure shock, so that to the extent that (2.8) is a good approximation for the end products of an explosion, the problem of the asymptotic limiting form of such a wave is subsumed here. This application is not, of course, of immediate practical significance, but is mentioned as of possible incidental interest in connection with previous work[4].

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3) Cf. BM-210, cited in footnote 2.

4) LA-143

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3. <u>The Lagrangian Equation of Motion</u>. One can now proceed immediately to substitute (1.2), (1.3), (2.7), (2.8) into the Eulerian equations of hydrodynamics for the conservation of mass, momentum, and entropy, thus deriving a system of three ordinary differential equations for the determination of p', v', u' (Cf. the paper of Guderley cited above). Alternatively, one can introduce a Lagrangian variable x - the radius of a laminar shell of the material when it is distributed at fixed constant density - and reduce the Lagrangian equation of motion to a single ordinary differential equation (Cf LA-210). The latter procedure is by far the less cumbersome, and leads more readily to a happy choice of variables; accordingly it is the one which we adopt. As our Lagrangian state of reference, we take the state of the material ahead of the shock.

To see that the Lagrangian formulation is equivalent to that of § 1, we start with the equation

$$v = y^2 x^{-2} y_x, \tag{3.1}$$

which, with (1.2), (1.3), $q_2 = 0$ gives

$$x^2 \, dx = a^3 v^{-1}(f) \, f^2 \cdot df \cdot t^{3n}$$

Thus

$$x^3 = F(f) t^{3n} + G(t),$$

and since on the shock $y = x$, $f = f_o$, we have $G(t) \sim t^{3n}$. Hence we are justified in introducing

$$x = a w t^n, \quad y = a f (w) t^n. \tag{3.2}$$

Now the function k in (2.8) depends only on x, not on t; so by (1.3), (2.7),

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we have

$$k(x) = Ax^{2(n-1)/n} = Aa^{2(n-1)/n}\, w^{2(n-1)/n}\, t^{2(n-1)}, \qquad (3.3)$$

where A is a constant over any region in the (x, t)-plane which contains no shocks, but can of course have different values in two such regions separated by a shock.

With (3.2), (2.8), (3.3), we can proceed to derive the governing ordinary differential equation. The Lagrangian equation of motion is

$$y_{tt} = -\rho_o^{-1} y^2 x^{-2} p_x. \qquad (3.4)$$

From (3.2), we have

$$\frac{\partial}{\partial x} = a^{-1} t^{-n} \frac{\partial}{\partial w}, \quad y_x = f', \quad y_t = an(f - wf')t^{n-1}, \qquad (3.5)$$

$$y_{tt} = an^2(w^2 f'' + mwf' - mf)t^{n-2}, \qquad (3.6)$$

where

$$m = (1 - n)/n. \qquad (3.7)$$

For the right side of (3.4), we obtain, using (3.5), (2.8), (3.3),

$$\rho_o^{-1} y^2 x^{-2} p_x = \rho_o^{-1} a^{-2m-1} A f^2 w^{-2} (f^{-2\gamma} f'^{-\gamma} w^{2(\gamma - m)})' t^{n-2}. \qquad (3.8)$$

Accordingly, with

$$A = n^2 a^{2+2m} \rho_o \qquad (3.9)$$

which simply fixes the unit of pressure in terms of A, (3.4) becomes

$$w^2 f'' + mwf' - mf = f^2 w^{-2} (f^{-2\gamma} f'^{-\gamma} w^{2(\gamma - m)})'. \qquad (3.10)$$

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For certain purposes, there is an advantage in the further change of variable

$$g(\sigma) = f\, w^{-1}, \quad \sigma = w^{-1} \quad . \tag{3.11}$$

From this we have

$$f' = g - \sigma g', \quad f'' = \sigma^3 g'' \tag{3.12}$$

and (3.10) becomes

$$\sigma g'' - m g' = g^2 \gamma^2 \left[g^{-2\gamma} (g - \sigma g')^{-\gamma} \sigma^{2m} \right]' \tag{3.13}$$

For future reference, we note the relations

$$u = a\, n (f - w f')\, t^{n-1} = a\, n\, g'\, t^{n-1} ,$$

$$v = f^2 w^{-2} f' = g^2 (g - \sigma g'), \tag{3.14}$$

$$p = n^2 a^2 \rho_0 w^{2(\gamma - m)} f^{-2\gamma} f'^{-\gamma} \; t^{2(n-1)} = n^2 a^2 \rho_0 g^{-2\gamma} (g - \sigma g')^{-\gamma} \sigma^{2m} t^{2(n-1)} .$$

In all the above equations involving t, the sense of time is reversed for the inward motion. Also, it is to be borne in mind that since A has two different values on the two sides of a shock, the same is true of a, by (3.9).

Equation (3.10), or alternatively (3.13), then, is the Lagrangian equation of motion, assuming similarity of the type described in § 1, and it is worthwhile to examine from a slightly different point of view the significance of this assumption. In the first place, the assumption that the shock has the form $y = a f_0 t^n$ may be regarded, not as an assumption at all, but merely as a focusing of attention on the term of lowest order in the expansion of the shock. In this sense, the only assumption is that y can be expanded in powers of t along the shock. Next, without reference to (1.3), the introduction of the variable w is certainly a natural procedure, since it has the effect of replacing the region in the (x,t)-plane bounded by $t = 0$

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and $x = af_0 t^n$, by a quadrant in the (w,t)-plane, and simply introduces instead of (3.4), an equivalent differential equation. Now a natural and familiar method for solving such a partial differential equation over such a region, is to assume an expansion of the form

$$y = a \sum_{j=1}^{\infty} f_j(w)\ t^{nj} \qquad (3.15)$$

and this leads in general to a sequence of ordinary differential equations for the functions f_j. In our case the equation (3.9) is simply the first equation of this sequence. Regarded in this light, then, the assumption of similarity is only an assumption regarding the regularity of the function y in its dependence on w, t^n, and thus one of the kind that is necessarily consistently adopted in physics. This, of course, says nothing with regard to the adiabatic equation of state (2.8), but even there a similar point of view may be adopted. For p is completely determined by S, v, and thus, for $v \sim 0$, $p \sim \infty$, we may reasonably assume

$$p = \sum_{j=1}^{\infty} k_j(S)\ v^{-\gamma j}$$

The equation (2.8) may then be taken as the first term in this expansion, and thus (2.8) is valid to the extent that only this first term is significant. Hence, irrespective of quantitative considerations and the gaps in our experimental knowledge, results derived from the procedure we have adopted do at the least furnish qualitative insight into the problem, and must describe the limiting behavior correctly to the extent that the hydrodynamical idealization itself has validity.

4. The Boundary Conditions. There are four curves in the (x,t)-plane which play a critical role in the problem before us, namely, the incoming shock, the line $t = 0$, the reflected shock, and the half-line $x = 0$, $t > 0$. We consider now

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the conditions to be satisfied on these.

(a) The converging shock: Here $y = x$, so $f = w$, or $g(\sigma) = 1$. Next the constant compression ratio on the shock is a parameter in the problem, and must be given before quantitative results can be determined. We denote its value by μ, as above. Finally, we must satisfy (2.6), which in view of (2.4) may be written

$$p = \rho_0 U^2 (1 - \mu) \tag{4.1}$$

Now, by (3.3), (3.4), $p = n^2 a^2 \rho_0 \mu^{-\gamma} \sigma^{2m} t^{2(n-1)}$ and $U^2 = a^2 n^2 \sigma^{-2} t^{2(n-1)}$.

Thus, (4.1) becomes

$$\sigma^{2+2m} = (1 - \mu)\mu^{\gamma} \tag{4.2}$$

Hence, using (3.14), we have on the converging shock

$$\sigma = \left((1 - \mu)\mu^{\gamma}\right)^{1/[2(1+m)]}, \quad g = 1, \quad g - \sigma g' = \mu \tag{4.3}$$

Note that since a is a free scalar, no point in the incoming shock is determined by (4.2).

The rôle of the equation (2.3) for the conservation of energy is this: It permits us to find E as a function of p, v, and the parameter μ, in the light of (2.8). For we have

$$dE = -pdv + TdS$$

(T = temperature), and thus, by (2.8)

$$E = \frac{pv}{(\gamma - 1)} + G(k) \tag{4.4}$$

Now, taking $G(k) = 0$ for $p = 0$, we have on the high pressure side of the shock, from (2.3)

$$E = \frac{1}{2} p (1 - \mu). \tag{4.5}$$

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Thus

$$G(k) = p\left[\frac{1}{2}(1-\mu) - \frac{1}{\gamma-1}\mu\right] = \frac{1}{2}p\left(1 - \frac{\gamma+1}{\gamma-1}\mu\right).$$

Hence

$$G(k) = Ck, \quad C = \frac{1}{2}\left(1 - \frac{\gamma+1}{\gamma-1}\mu\right)\mu^{-\mu}, \tag{4.6}$$

and

$$E = \frac{pv}{\gamma-1}\left[1 + (\gamma-1)\,Cv^{\gamma-1}\right]. \tag{4.7}$$

For an ideal gas, of course, $C = 0$, and one has necessarily the familiar result $\mu = (\gamma - 1)/(\gamma + 1)$; for the other applications mentioned, however, this value of μ is not necessarily prescribed.

(b) $t = 0$: Here, for fixed x, $w \to \infty$, $f(w) \to \infty$, $\sigma \to 0$. However, y, y_t must be regular functions of x. Since, always, $y = g(\sigma)x$, it follows that $g(\sigma) = C_o > 0$, and since $y_t = ang' t^{n-1} = a^{1/n} g' \sigma^{-m} x^{-m}$, we have $g' \sim \sigma^m$ for $\sigma \to 0$. Thus, at $\sigma = 0$

$$g(\sigma) = C_o + C_1\sigma^{1+m} + \text{------}, \quad C_o > 0, \quad C_1 \neq 0\ . \tag{4.8}$$

It is readily determined that the general solution of (3.12), in the neighborhood of $\sigma = 0$ is

$$g(\sigma) = \sum_{j=0}^{\infty} C_j\,\sigma^{j(1+m)} \tag{4.9}$$

with C_o, C_1 as constants of integration, $C_j = C_j(C_o, C_1; m)$, $j > 1$. Since, for the inward motion, the sense of time is reversed, the continuity of y, u at $t = 0$ implies that C_o is the same for the inward and outward motion, while the values of C_1 for the two phases differ only in sign, $C_1 > 0$ (inward), $C_1 < 0$ (outward).

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(c) The reflected shock: Since A, a have different values on the two sides of this shock, while x, y, t are of course continuous, f, w are discontinuous; $a_1 f_1 = a_2 f_2$, $a_1 w_1 = a_2 w_2$. Accordingly σ is discontinuous, g continuous:

$$a_1 \sigma_2 = a_2 \sigma_1 \ , \ g_1 = g_2. \tag{4.10}$$

From (2.1), (2.2),

$$v_1^{\,2}(p_2 - p_1) = \rho_o (v_1 - v_2)(U - u_1)^2.$$

Now, $U - u_1 = a_1 \, n \, w_1 \, f_1 \, t^{n-1}$, $v_1 = g^2 f_1{}'$,

so we have

$$g^2(p_2 - p_1) = n^2 a_1^{\,2} \rho_o \sigma_1^{-2}(v_1 - v_2) t^{2(n-1)}$$

Also

$$p_1 = a_1^{\,2} \sigma_1^{\,2m} v_1^{-\gamma} t^{2(n-1)} = a_1^{\,2} \sigma_1^{-2} \sigma_1^{\,2+2m} v_1^{-\gamma} \ t^{2(n-1)},$$

and using (4.10),

$$p_2 = a_2^{\,2} \sigma_2^{\,2m} v_2^{-\gamma} \ t^{2(n-1)} = a_1^{\,2} \sigma_1^{-2} \sigma_2^{\,2+2m} v_2^{-\gamma} \ t^{2(n-1)}.$$

Thus, (2.2) becomes

$$g^4(\sigma_2^{\,2+2m} v_2^{-\gamma} - \sigma_1^{\,2+2m} v_1^{-\gamma}) = v_1 - v_2 \ , \tag{4.11}$$

or

$$g^{1-3\gamma}\left[\sigma_2^{\,2+2m}(g - \sigma_2 g_2{}')^{-\gamma} - \sigma_1^{\,2+2m}(g - \sigma_1 g_1{}')^{-\gamma}\right] = g_2{}' - g_1{}' . \tag{4.12}$$

Finally, from (4.7), (2.3), (4.10)

$$\left(\frac{\sigma_2}{\sigma_1}\right)^{2+2m}\left(\frac{v_1}{v_2}\right)^{\gamma} = \frac{(\gamma + 1)v_1 - (\gamma - 1)v_2 + 2(\gamma - 1)Cv_1^{\gamma}}{(\gamma + 1)v_2 - (\gamma - 1)v_1 + 2(\gamma - 1)Cv_2^{\gamma}} \tag{4.13}$$

(d) $x = 0$, $t > 0$: Here we have $w = 0$, $f(w) = 0$, so it is convenient to use (3.10), rather than (3.13). Setting

$$f = A_\beta w^\beta \tag{4.14}$$

we obtain for the term of lowest degree on the left side of (3.10),

$$A_\beta(\beta - 1)\ (\beta + m)\ w^\beta ,$$

and for the term of lowest degree on the right-hand side

$$-\beta^{-\gamma} A_\beta^{2-3\gamma}\left[3\gamma(1-\beta) - 2m\right] w^{(2-3\gamma)\beta + 3(\gamma-1) - 2m} .$$

Now, if $\beta = 1$, the left side contributes only terms of degree greater than 1, while the term of lowest degree on the right has degree $-1 - 2m$, obviously less than 1. So $\beta = 1$ is impossible, and $\beta = -m$ is obviously absurd. Hence we have

$$\beta = (3\gamma - 2m)/3\gamma , \quad A_\beta \text{ arbitrary;} \tag{4.15}$$

or the two above expressions are of the same degree and

$$\beta = \frac{3(\gamma-1) - 2m}{3\gamma - 1} , \quad A_\beta^{1-3\gamma} = \frac{3(\gamma-1)(1+m)^2\beta^\gamma}{(3\gamma-1)(3\gamma+m)} . \tag{4.16}$$

In both cases $\beta < 1$, and since $v \sim w^{3(\beta-1)}$, $w \to 0$, we have $v \to \infty$ in both cases. On the other hand $p \sim w^{-2m} v^{-\gamma} \sim w^{3\gamma - 2m - 3\gamma\beta}$. So, according as (4.15) or (4.16) holds, we have $p \sim$ const., or $p \sim w^{(6\gamma + 2m)/(3\gamma - 1)} \to 0$ for $w \to 0$. We shall see in § 9 that (4.15) is actually the only possibility, since the boundary conditions (c) cannot in fact be satisfied for (4.16).

5. <u>Thermodynamic Considerations</u>. The problem outlined in §§ 3,4 is completely determinate from the point of view of dynamics. Thermodynamically, however, it is indeterminate without the postulation of a further relation - for example, specification of the dependence of k on entropy would suffice. However, even

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without this (2.8), (4.7) have certain thermodynamic implications, and we pause to consider these. Using (2.8), we have from (4.7),

$$T = \left(\frac{dE}{dS}\right)_v = \frac{1}{\gamma - 1}\left(v^{-\gamma + 1} + (\gamma - 1)C\right)\frac{dk}{dS}. \tag{5.1}$$

Now, certainly $dk/dS > 0$, so we must have

$$v^{-\gamma + 1} + (\gamma - 1)\, C \geqq 0 \tag{5.2}$$

But we have just seen that at $x \to 0$, $t > 0$, $v \to \infty$, and thus (5.2) reduces to

$$C \geqq 0, \qquad \mu \leqq (\gamma - 1)/(\gamma + 1). \tag{5.3}$$

There are two points of view which one can adopt with respect to (5.3). First, if one regards (2.8) as universally valid, then (5.1) is also universally valid and (5.3) is a necessary condition on μ. On the other hand, if one takes (2.8) to be an approximation valid only for $v \ll \infty$, then (5.3) has no special significance. In this instance however, (2.8) must be replaced by something else behind the reflected shock, and one is not in position to consider the reflection problem at all, without knowledge of the adiabatics for $v \sim \infty$. So (5.3) must certainly be satisfied if the problem of the reflected shock is to be considered by the method adopted in §§ 3,4.

By way of further orientation, let us suppose that

$$T = h(v)\, p^{\beta}. \tag{5.4}$$

This is justifiable, insofar as we are interested in $p \sim \infty$, $T \sim \infty$. Then, from (5.1), (5.4)

$$h(v)\, v^{-\gamma\beta} k^{\beta} = \frac{1}{\gamma - 1}\left[v^{-\gamma + 1} + (\gamma - 1)C\right]\frac{dk}{dS}.$$

Since k is a function of S alone, this yields

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$$h(v) = D\left[v^{-\gamma(1-\beta)+1} + (\gamma - 1)\,Cv^{\gamma\beta}\right], \quad D = \text{const.} > 0.$$

$$\frac{dk}{dS} = Dk^{\beta}\,.$$

If $\beta = 1$, we have

$$k = Ke^{DS}, \tag{5.5}$$

otherwise

$$k = (cS + d)^{1/(1-\beta)} \tag{5.6}$$

Since, for $S \longrightarrow \infty$, $k \longrightarrow \infty$, we have $\beta \leqq 1$. Now, from

$$Dp^{\beta}\left(v^{-\gamma(1-\beta)+1} + (\gamma - 1)Cv^{\gamma\beta}\right) = T$$

we have

$$\bigl(1 - \gamma(1-\beta)\bigr) + \gamma\beta\,(\gamma - 1)Cv^{\gamma-1} \geqq 0 \tag{5.7}$$

since for T = const., p must be monotone decreasing in v. This is always satisfied if $\beta \geqq (\gamma - 1)/\gamma$, but can fail for smaller β. Thus, for such β, (5.7) can result in a stronger restriction on μ. Or, to put it another way, to assume a fixed value for μ is inconsistent with too small a β in (5.4), i.e., implies $\beta \geqq \beta_0$. The value of β_0, however depends in general on the range of v in the problem, and has as we have observed the value $(\gamma - 1)/\gamma$ only for $\mu = (\gamma - 1)/(\gamma + 1)$. We do not pursue this matter further, but mention it simply by way of emphasizing that the choice of μ imposes restrictions on any thermodynamic assumptions which one may wish to make subsequently.

6. Transformation of the Fundamental Equation to an Equation of the First Order.

We now apply to (3.13) the technique used in LA-210. The analytical details

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are so closely similar to those of the report cited that we need only outline the procedure and give the results. The first step is to introduce new variables s, h, ℓ, defined by the equations

$$\sigma = e^{s}, \quad g = e^{as}h, \quad g - \sigma g' = e^{as}\ell \,. \tag{6.1}$$

From the second and third equations we have

$$\left[(1 - a)h - \ell\right] \, ds = dh. \tag{6.2}$$

Substituting (6.1) in (3.13), we find that the choice

$$a = \frac{2(1 + m)}{3\gamma - 1} \tag{6.3}$$

renders the equation homogeneous in σ, and using (6.2), we obtain accordingly

$$\begin{aligned} &\left[(1 - a)h - \ell\right] \left[1 - \gamma h^{2 - 2\gamma} \ell^{-(\gamma + 1)}\right] d\ell = \\ &= \left\{2\gamma h^{1 - 2\gamma} \ell^{-\gamma} \left[(1 - a)h - \ell\right] - \left[(a - m)\ell + mh - (a+2)h^{2-2\gamma}\ell^{-\gamma}\right]\right\} dh \,. \end{aligned} \tag{6.4}$$

Next, in order to get rid of the γ in the exponents, we set

$$\xi = \ell h^{-1}, \quad \eta = h^{1-2\gamma}\ell^{-\gamma} \,. \tag{6.5}$$

Then (6.4) becomes simply

$$\frac{d\eta}{d\xi} = \frac{\eta}{\xi} \, \frac{N}{D}, \tag{6.6}$$

where

$$\left.\begin{aligned} N &= (2\gamma - 1)\xi^2 + \gamma\eta\xi - \left(2\gamma - 3 + (\gamma - 2)m\right)\xi - 3\gamma\eta + \gamma m, \\ D &= \xi^2 - 3\gamma\eta\xi - (1 - m)\xi + (3\gamma - 2m)\eta - m. \end{aligned}\right\} \tag{6.7}$$

One obtains readily the relation

$$N + \gamma D = (3\gamma - 1)(\xi - 1 + a)(\xi - \gamma\eta), \tag{6.8}$$

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the importance of which will appear presently, and finally, from (6.2), (6.4), (6.5), we have

$$ds = -\frac{\gamma\eta - \xi}{D}\,\frac{d\xi}{\xi} = -\frac{\gamma\eta - \xi}{N}\,\frac{d\eta}{\eta}\,. \tag{6.9}$$

For subsequent reference, we now note various relations between the variables ξ, η, and the variables, g, σ, u, v, p:

$$\left.\begin{aligned}\xi &= 1 - g^{-1}\,g'\sigma\,,\\ \eta &= g^{1-2\gamma}\,(g - \sigma g')^{-\gamma}\,\sigma^{2(1+m)}\,;\end{aligned}\right\} \tag{6.10}$$

$$g = \eta^{-1/(3\gamma - 1)}\,\xi^{-\gamma/(3\gamma - 1)}\,\sigma^{\alpha}\,; \tag{6.11}$$

$$\left.\begin{aligned}v &= g^3\xi\,,\quad u = ang\sigma^{-1}(1 - \xi)t^{n-1}\,,\\ p &= n^2 a^2 \rho_0\, g^{-1}\sigma^{-2}\eta\, t^{2(n-1)}\,.\end{aligned}\right\} \tag{6.12}$$

Next, we formulate the boundary conditions (a), (b), (c), (d) of §4 in terms of ξ, η, s: (a) The converging shock: Substitution of (4.3) in (6.10) yields at once

$$\xi = \mu,\quad \eta = 1 - \mu,\quad s = \frac{1}{2(1+m)}\,\log\,(1-\mu)\mu^{\gamma} \tag{6.13}$$

(b) t = 0: Here $\sigma \rightarrow 0$, so $s \rightarrow -\infty$. Substituting (4.8) in (6.10), moreover, gives

$$\left.\begin{aligned}\xi &= 1 - (1+m)C_0^{-1}\,C_1\,e^{(1+m)s} + \ldots\,,\\ \eta &= C_0^{\,1-3\gamma}\,e^{2(1+m)s} + \ldots\,.\end{aligned}\right\} \tag{6.14}$$

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Thus, $\xi = 1$, $\eta = 0$, for $s = -\infty$. Moreover, we have, from (6.14)

$$\frac{d\xi}{d\eta} = -\frac{(1+m)}{2} C_o^{3(\gamma - 1)/2} C_1 \eta^{-1/2} + \ldots,$$

Thus, summing up,

$$s \longrightarrow -\infty, \quad \xi \longrightarrow 1, \quad \eta \longrightarrow 0, \quad \xi = 1 - B\eta^{1/2} + \ldots, \qquad (6.15)$$

where

$$B = (1 + m) C_o^{3(\gamma - 1)/2} C_1 . \qquad (6.16)$$

We have already observed that the C_o is the same for the inward and outward motions, while the two C_1's differ in sign, the C_1 for the inward motion being positive. Hence $B > 0$, for the inward motion, and its negative belongs to the outward motion. (c) The reflected shock: From (6.10), we have

$$\eta \xi^{\gamma} = g^{1 - 3\gamma} \sigma^{2(1+m)} .$$

Hence, since g is continuous,

$$\sigma_2^{2(1+m)} \eta_1 \xi_1^{\gamma} = \sigma_1^{2(1+m)} \eta_2 \xi_2^{\gamma} . \qquad (6.17)$$

Using the first of the equations (6.12), with (4.11) we now obtain

$$g^{1 - 3\gamma}(\sigma_2^{2+2m} \xi_2^{-\gamma} - \sigma_1^{2+2m} \xi_1^{-\gamma}) = \xi_1 - \xi_2 ,$$

and this with $\eta \xi^{\gamma} = g^{1-3\gamma} \sigma^{2+2m}$ gives

$$\xi_1 + \eta_1 = \xi_2 + \eta_2 . \qquad (6.18)$$

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Finally, (4.13) can be written

$$\left(\frac{\sigma_2}{\sigma_1}\right)^{2(1+m)}\left(\frac{\xi_1}{\xi_2}\right)^{\gamma} = \frac{(\gamma+1)\xi_1 - (\gamma-1)\xi_2 + 2(\gamma-1)cg^{3(\gamma-1)}\xi_1^{\gamma}}{(\gamma+1)\xi_2 - (\gamma-1)\xi_1 + 2(\gamma-1)cg^{3(\gamma-1)}\xi_2^{\gamma}} .$$

Combining this with (6.17), we obtain

$$\frac{\eta_2}{\eta_1} = \frac{(\gamma+1)\xi_1 - (\gamma-1)\xi_2 + 2(\gamma-1)cg^{3(\gamma-1)}\xi_1^{\gamma}}{(\gamma+1)\xi_2 - (\gamma-1)\xi_1 + 2(\gamma-1)cg^{3(\gamma-1)}\xi_2^{\gamma}} . \qquad (6.19)$$

Now, let the subscripts 2 denote the high-pressure, and thus high-density side of the shock. Then, since $\rho_2 - \rho_1 \sim \xi_1 - \xi_2$, we have, using also (6.18)

$$\xi_2 < \xi_1 , \quad \eta_2 > \eta_1 . \qquad (6.20)$$

It is to be observed that the conditions (6.18) (6.19), (6.20), while of special interest to us with respect to the reflected shock, are in reality entirely general and apply to any shock in the material after the initial shock, however induced. They imply, incidentally, that such a shock must be on a similarity curve $y = af(w)t^n$, and thus render our assumption in § 1 that the reflected shock is such a curve, a theorem.

For a given ξ_1, η_1, g, the corresponding ξ_2, η_2, are obtained by intersecting the straight line (6.18) with the curve (6.19). One point of intersection is $\xi_2 = \xi_1$, $\eta_2 = \eta_1$, and the second is the desired one. That there are only two is apparent from the fact that for any σ_2, the curve (6.17), and (6.18) have at most two points of intersection. To determine when there is only one point of intersection, we set

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$$\xi_2 = \xi_1 + \varepsilon$$

Then, (6.18) gives $\eta_2 = \eta_1 - \varepsilon$, and (6.19) becomes

$$\eta_1 = \gamma^{-1}\xi_1 + O(\varepsilon)$$

Hence the high-pressure side of the shock is always above, the low-pressure side always below, $\gamma\eta - \xi = 0$, i.e.

$$\gamma\eta_2 - \xi_2 > 0, \quad \gamma\eta_1 - \xi_1 < 0 . \qquad (6.21)$$

Thus as the strength of the shock approaches zero $(\xi_1\eta_1)$, $(\xi_2\eta_2)$ tend to coincidence on the line $\gamma\eta - \xi = 0$. This is in fact to be expected since $\gamma\eta - \xi = 0$ is one of the conditions for a characteristic through the origin, and thus a necessary condition for a sound wave through the origin. To see this we have only to note that in such a characteristic the coefficient of g'' in (3.13) must vanish. This coefficient is readily found to be $\sigma\left[1 - \gamma g^{2 - 2\gamma}\sigma^{2 + 2m}(g - \sigma g')^{-\gamma - 1}\right]$ which by (6.10) is $\sigma(1 - \gamma\eta\xi^{-1})$. Thus it vanishes for $\gamma\eta - \xi = 0$ and $\sigma = 0$. The latter root corresponds to $x = \infty$, $p = 0$, sound velocity $= 0$, so the significant one is $\gamma\eta - \xi = 0$.

(d) The line $x = 0$, $t > 0$: Here $w \to 0$, $\sigma \to +\infty$, $s \to +\infty$. From (4.14), we have

$$g = A_\beta \sigma^{1-\beta} + \ldots, \quad g - \sigma g' = \beta A_\beta \sigma^{1-\beta} + \ldots .$$

Hence, by (6.18),

$$\xi \to \beta, \quad \eta = \beta^{-\gamma} A_\beta^{1-3\gamma} \sigma^{(3\gamma - 1)\beta - 3\gamma + 3 + 2m} + \ldots$$

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Hence, if (4.15) holds, we have

$$\xi \longrightarrow \frac{3\gamma - 2m}{3\gamma}, \quad \eta \rightarrow \infty \tag{6.22}$$

while if (4.16) holds, we have

$$\xi \longrightarrow \frac{3(\gamma - 1) - 2m}{(3\gamma - 1)}, \quad \eta \rightarrow \frac{3(\gamma - 1)(1 + m)^2}{(3\gamma - 1)(3\gamma + m)} \,. \tag{6.23}$$

It is convenient now to introduce as follows symbols for the points in the (ξ, η)-plane which correspond to our boundary conditions:

$$\left.\begin{array}{l} P_\mu : \xi = \mu, \quad \eta = 1 - \mu ; \quad P_1 : \xi = 1, \ \eta = 0 ; \\ P_\infty : \xi = \dfrac{3\gamma - 2m}{3\gamma}, \quad \eta = \infty ; \\ P_o : \xi = \dfrac{3(\gamma - 1) - 2m}{3\gamma - 1}, \quad \eta = \dfrac{3(\gamma - 1)(1 + m)^2}{(3\gamma - 1)(3\gamma + m)} . \end{array}\right\} \tag{6.24}$$

Our required solution can then be described as follows: It is composed of two integral curves of (6.6); one a curve $P_\mu P_1 P_{S_1}$, where P_{S_1} is the point (ξ_1, η_1) corresponding to the low-pressure side of the reflected shock; the other a curve $P_{S_2} P_\infty$, or $P_{S_2} P_o$ where P_{S_2} is the point (ξ_2, η_2) corresponding to the high-pressure side of the reflected shock. The points (ξ_1, η_1), (ξ_2, η_2) are related by (6.18), (6.19). The curve $P_\mu \ P_1 \ P_{S_1}$ must have $d\eta/d\xi = 0$ at P_1, and on $P_\mu \ P_1$, s must be monotone decreasing from its value at P_μ to $-\infty$ at P_1. On the discontinuous curve $P_1 \ P_{S_1} - P_{S_2} \ P_\infty$ (P_o) S must be monotone increasing from $-\infty$ at P_1 to $+\infty$ at $P_\infty (P_o)$. In order to understand at least qualitatively the implications of all of these conditions, we turn next to an examination of the vector field of (6.6).

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7. <u>Qualitative Discussion of the Reduced Differential Equation.</u> To begin, we consider the implications of the required monotonicity of S on the curve $P_\mu P_1$. For this we must have

$$\frac{\gamma\eta - \xi}{D}\,\frac{d\xi}{\xi} = \frac{\gamma\eta - \xi}{N}\,\frac{d\eta}{\eta} \geqq 0\,, \tag{7.1}$$

by (6.9). From (6.12), $\eta < 0$ implies $p < 0$, while $\xi < 0$ implies $v < 0$, so always $\xi \geqq 0$, $\eta \geqq 0$. From the equation (6.6) itself, it follows that a change in sign in $d\xi$, $(d\eta)$ implies and is implied by a change in sign in $D(N)$, except at singularities of the differential equation. Hence, what matters is whether or not $P_\mu P_1$ crosses $\gamma\eta - \xi = 0$. Now P_μ is on the line $\xi + \eta = 1$, and this intersects $\gamma\eta - \xi = 0$ at $\xi = \gamma/(\gamma + 1)$, $\eta = 1/(\gamma + 1)$. Hence if (5.3), $\mu \leqq (\gamma - 1)/(\gamma + 1)$, is to be satisfied, P_μ lies above $\gamma\eta - \xi = 0$, and P_1 is always below it. Even without (5.3), in fact, the point P_μ must lie above $\gamma\eta - \xi = 0$. For from (6.12), we have for sound velocity c relative to the material

$$c^2 = \gamma p \rho^{-1} = \gamma n^2 a^2 g^2 \sigma^{-2} \eta \xi\, t^{2(n-1)}$$

so

$$u + c = a\,n\,g\,\sigma^{-1}\left[1 - \xi + \sqrt{\gamma\eta\xi}\right] t^{n-1}$$

Hence $U/(u + c) = (1 - \xi + \sqrt{\gamma\xi\eta}\,)^{-1}$, and this is less or greater than unity according as $\gamma\eta \gtrless \xi$. Hence, without reference to (5.3),

$$\mu < \frac{\gamma}{\gamma + 1} \tag{7.2}$$

since a shock with velocity less than sound velocity behind it could not be started. This needs to be noted in any problem in which (2.8) is not taken to be universal.

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From (7.1), (6.8) it follows that the solution $P_\mu P_1$ must cross $\gamma\eta - \xi = 0$ at a point of intersection with $N = 0$, $D = 0$, and hence we must insist on the existence of such a point. Substituting $\gamma\eta = \xi$ in $N = 0$, we obtain the two roots

$$\xi_{\pm} = \frac{2\gamma + (\gamma - 2)\, m \pm \sqrt{[2\gamma + (\gamma - 2)m]^2 - 8\gamma^2 m}}{4\gamma} \tag{7.3}$$

and thus have

$$[2\gamma + (\gamma - 2)m]^2 - 8\gamma^2 m \geq 0,$$

which is equivalent to

$$m \leq 2\gamma\,(\sqrt{\gamma} + \sqrt{2})^{-2}, \text{ or } m \geq 2\gamma(\sqrt{\gamma} - \sqrt{2})^{-2}, \tag{7.4}$$

as a condition on m. It is readily determined from (7.3), moreover, that for $m \geq 2\gamma\,(\sqrt{\gamma} - \sqrt{2})^{-2}$, we have $\xi_{\pm} < 0$ or $\xi_{\pm} > 1$ according as $\gamma \lessgtr 2$. Now $\xi_{\pm} > 1$ means $u < 0$, and on $P_\mu P_1$ this would mean that the material is moving out rather than in, while $\xi_{\pm} < 0$ means, as we have noted, negative v. Hence

$$m \leq 2\gamma(\sqrt{\gamma} + \sqrt{2})^{-2} \tag{7.5}$$

It is now helpful to examine the configuration of curves $N = 0$, $D = 0$, $\gamma\eta - \xi = 0$, $\xi + \eta = 1$ in a few typical cases. These are shown in Figs. 1a, 1b, 1c, for $\gamma = 1.4$, $m = .4$; $\gamma = 3$, $m = .45$; $\gamma = 8$, $m = .8$; respectively. The signs in the various portions of the plane separated by $N = 0$, $D = 0$ are the signs of du/dv in those regions.

Our next step is to analyze qualitatively the vector field of the differential equation. To this end, we begin by considering its behavior in the neighborhood of its singularities, i.e., points of intersection of the

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loci, $uN = 0$, $vD = 0$. In the portion of the plane in which we are interested, these points are as follows:

(0): the origin: $\xi = 0$, $\eta = 0$;

(1): P_1 : $\xi = 1$, $\eta = 0$;

(2): P_2: $\xi = 0$, $\eta = m/3$;

(3): the points P_0, P_-, P_+.

As in LA-210, excluding the special cases of coincidence of one or more of these points, the possibilities for any one of them is that it be hyperbolic, a spiral point, or a sheaf-point. The reader who is unfamiliar with the meaning of these terms, or with the analysis of singularities of first-order differential equations, is referred to § 5 of LA-210, where a brief discussion is given. We assume here complete familiarity with that material and turn now to a discussion of the points listed above.

(0): $\xi = 0$, $\eta = 0$. This point plays no direct rôle in our problem, but it is worthwhile to note its nature. It is readily seen to be a hyperbolic point, the two solutions through it being $\xi = 0$, $\eta = 0$.

(1): P_1: $\xi = 1$, $\eta = 0$. This is the point corresponding to $t = 0$, as we have noted in § 6, (b). Setting $\eta = \lambda(\xi - 1) + \ldots$, in (6.6), we find the integral

$$I_1: \eta = -\frac{1+m}{2m}(\xi - 1) + \ldots, \tag{7.6}$$

and the integral

$$\eta = B^{-2}(\xi - 1)^2 + \ldots \tag{7.7}$$

where B is arbitrary and related to C_0, C_1, by (6.16), according to § 6, (b).

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Fig. 2 shows the disposition of solutions in the neighborhood of P_1. The arrows on integral curves denote the direction of decreasing s.

(2): P_2 : $\xi = 0$, $\eta = m/3$. This point is hyperbolic, one obvious solution being $\xi = 0$. To find the other, we proceed as above, setting

$$\eta = m/3 + \lambda \xi + \ldots,$$

in (6.6). The result is the integral

$$I_2 : \eta = \frac{m}{3}\left[1 - \frac{2(\gamma - 3)m + 3(2\gamma - 3)}{9\gamma + 3(\gamma - 1)m - 2m^2} \xi + \ldots \right. \tag{7.8}$$

Note that here λ is negative for $\gamma \geqq 3$, positive for $\gamma \leqq 3/2$, and may have either sign for intermediate values. The disposition of solutions for the two cases are shown in Figs. 3a, and 3b. The arrows have the same significance as above.

(3): The points P_0, P_-, P_+. Let (ξ^*, η^*) be the co-ordinates of one of these points, and let λ_N, λ_D be respectively the slopes of N = 0, D = 0 at the point. Let

$$\xi = \xi^* + \xi', \quad \eta = \eta^* + \eta' \tag{7.9}$$

and let

$$\eta' = \lambda \xi' + \ldots \quad . \tag{7.10}$$

Then, from (6.6), (6.7)

$$\lambda = -\frac{\eta^*}{\xi^*} \frac{\gamma(3 - \xi^*)(\lambda - \lambda_N)}{(3\gamma - 2m - 3\gamma\xi^*)(\lambda - \lambda_D)} ,$$

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or

$$(3\gamma - 2m - 3\gamma\xi^*)\lambda^2 + \left[\frac{\gamma\eta^*}{\xi^*}(3 - \xi^*) - (3\gamma - 2m - 3\gamma\xi^*)\lambda_D\right]\lambda - \frac{\gamma\eta^*}{\xi^*}(3 - \xi^*)\lambda_N = 0 \qquad (7.11)$$

Since $\xi = (3\gamma - 2m)/3\gamma$ is the vertical asymptote of $D = 0$, we have $\xi < (3\gamma - 2m)/3\gamma$ at each of these points and thus at each point where $\lambda_N \geqq 0$, certainly the roots of (7.11) are real and it is not a spiral point. Now at $\xi = 0$, $N = 0$ lies above $\gamma\eta - \xi = 0$, and hence at P_+, the second intersection of these two curves, $N = 0$ crosses $\gamma\eta - \xi = 0$ from below to above; hence at P_+, $\lambda_N > 0$ and P_+ is never spiral. Moreover, for the minimum on $N = 0$, we find the ξ co-ordinate

$$\xi_{min.} = 3 - \left[\frac{12\gamma - 2(\gamma - 3)m}{2\gamma - 1}\right]^{1/2} \qquad (7.12)$$

and for m satisfying (7.5) this is always less than $\xi_0 = (3\gamma - 3 - 2m)/(3\gamma - 1)$. So P_0 always lies to the right of the minimum, $\lambda_N > 0$ there, and it is not a spiral point. On the other hand, λ_N can have either sign at P_-, so this point may be a spiral point.

Now consider the point of the three farthest to the right; since $P_- \leqq P_+$, this will be either P_0 or P_+. Since the curves $N = 0$, $D = 0$ have, respectively, the vertical asymptotes $\xi = 3$, $\xi = (3\gamma - 2m)/3\gamma$, $D = 0$ lies above $N = 0$ to the right of this point. Now consider any point to the right of and above the point, to the right of $D = 0$, and to the left of $N = 0$. Then the integral curve of (6.6) through this point has positive slope, and so cannot cross $N = 0$ or $D = 0$. Hence it goes to the point P_0 or P_+ itself, and the point must be a sheaf-point. On the other hand, at the next farthest

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point to the right, i.e., the middle one of the three points (which can be either P_-, P_+, or P_o), $N = 0$ crosses $D = 0$ from below to above as ξ increases, and it is readily seen that this implies that the point is hyperbolic. Finally, at the point farthest to the left the crossing is similar to that at the point farthest to the right, and this point is therefore a sheaf-point if it is P_+, a sheaf-point or a spiral point if it is P_-. From these conclusions, we can get a qualitative picture of the integral curves of (6.6) in the region of the plane containing P_-, P_+, P_o, and this is shown in Fig. 4, for the ordering P_-, P_o, P_+, for a case in which P_- is not spiral. As before the arrows denote the direction of decreasing s. The nature of the other cases, i.e., those in which the middle point is P_- or P_+, is readily perceived as follows: Interchange the meanings of P_o and $P_-(P_+)$ in Fig. 4, draw as $\gamma\eta - \xi = 0$ a line through the appropriate pair, and change the arrows to correspond.

The integral curves marked I_-, I_o, I_+ are obtained by substituting the algebraically smaller of the roots of (7.11) in (7.10) and continuing the expansion. In every case, only integral powers of ξ' will appear. If the algebraically larger root is taken, then at the middle point, again only integral powers will appear and (7.10) will be the curve through the points P_o, P_-, P_+. At the other two points however, the expansion with the larger root will yield non-integral powers, the smallest less than 2, and this term will have a free coefficient, thus giving the multiplicity of solutions.

Finally, the special cases in which two or more of the points $P_{\pm}$, P_o coincide have not been discussed but the configuration of integral curves in

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those instances is readily imagined.

8. The Inward Motion; Dependence of the Exponent on the Compression Ratio of the Converging Shock. The problem of the inward motion can now be put as follows: For a given μ_0 we require the value or values of m for which an integral curve $P_\mu P_\pm P_1$ exists and has the following properties:

(1) it crosses $\gamma\eta - \xi = 0$ only at $P_\pm$;

(2) it does not go through P_0;

(3) at P_1 it lies below I_1.

From § 7 we see that there are always infinitely many integral curves through one of the points P_-, P_+, and in general at least two through the other. This is a reflection of the fact that $\gamma\eta - \xi = 0$ is one of the conditions for a characteristic through the origin, and in fact P_-, P_+ correspond to curves $y = af_\pm t^n$ which are characteristics. This can be proved directly; alternatively, however, the existence of two values for du/dv is in itself proof. For it implies the possibility of a discontinuity, for example, in $\partial p/\partial y$ there, and, since p, u, v are continuous, this would correspond to a sound wave. We have noted also that at $P_\pm$, we have in general $d^2u/dv^2 = \pm\infty$, and this gives $\partial^2p/\partial y^2 = \mp\infty$, again a sound wave. On the other hand, as we have observed in § 7, there are, if the two roots of (7.11) are distinct, two solutions through P_- and P_+ which are analytic there. Thus, according as our solution coincides with one of these or not, a sound wave converging to the origin behind the shock wave will not or will be present in the solution obtained.

The condition (2) has not been explicitly noted before, but from (6.9) it is evident that P_0 is a pole of s and so must be excluded.

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From Figs. 1 and 4, it is at once evident that solutions exist for some pairs (γ,μ), but we want to discuss the problem systematically. To begin, we shall disregard the restriction (5.3) on μ, and observe only the weaker condition (7.2). Apart from the fact that under the weak interpretation of (2.8) only (7.2) is significant there are analytic advantages in studying the problem in the neighborhood of $\gamma/(\gamma + 1)$.

In particular, if we consider the limiting case when the shock is exactly sonic and $\mu = \gamma/(\gamma + 1)$, then the point P_- or P_+ through which the solution goes must lie exactly on the line $\xi + \eta = 1$, and $\xi_{\pm} = \gamma/(\gamma + 1)$. Combining this with (7.3) we obtain

$$\frac{\gamma - 1}{2(\gamma + 1)} - \frac{(\gamma - 2)m}{4\gamma} = \frac{\pm\sqrt{\left[2\gamma + (\gamma - 2)m\right]^2 - 8\gamma^2 m}}{4\gamma} \quad . \tag{8.1}$$

For $\gamma \leq 2$, the left side of (8.1) is always positive, while for $\gamma > 2$, it is negative or zero only for $m \geq 2\gamma(\gamma - 1)/(\gamma + 1)(\gamma - 2)$. By (7.4) this is only possible if

$$\frac{\gamma - 2}{(\sqrt{\gamma} + \sqrt{2})^2} \geq \frac{\gamma - 1}{\gamma + 1}$$

or

$$\frac{\sqrt{\gamma} - \sqrt{2}}{\sqrt{\gamma} + \sqrt{2}} \geq \frac{\gamma - 1}{\gamma + 1}$$

which is equivalent to $\gamma \leq 1/2$. So (8.1) only has a solution for the positive radical, that is we can only have $\xi_+ = \gamma/(\gamma + 1)$. Solving (8.1), we find

$$m = \frac{2\gamma}{3(\gamma + 1)} \tag{8.2}$$

Now the above argument which excludes $\xi_- > \gamma/(\gamma + 1)$, also in fact excludes $\xi_- > \gamma/(\gamma + 1)$, while for $\xi_+ = \gamma/(\gamma + 1) = \mu$, $P_+ = P_\mu$, we have

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m given by (8.2). We now propose to show that for a solution $P_\mu P_+ P_1$, $\mu < \gamma/(\gamma + 1)$, P_+ must be below $\xi + \eta = 1$. To this end we substitute $\xi = 1 - \eta$ in (6.6), obtaining

$$\frac{d\eta}{d\xi} = \frac{(\gamma - 1)\xi^2 + [2\gamma + 3 - (\gamma - 2)m]\xi - \gamma(3 - m)}{\xi[3\gamma - 3m - (3\gamma + 1)\xi]} . \qquad (8.3)$$

Now, in the positive ξ-direction, solutions cross $\xi + \eta = 1$ from above to below, or below to above, according as (8.3) is less than or greater than -1, or according as

$$\frac{(\gamma + 1)\xi - \gamma}{3\gamma - 3m - (3\gamma + 1)\xi} \lessgtr 0 , \qquad (8.4)$$

as is readily shown. Here the root of the numerator is the intersection of $\xi + \eta = 1$ with $\gamma\eta - \xi = 0$ when (8.3) has the value -1, while the root of the denominator is its intersection with $D = 0$, where (8.3) has the value ∞. So, in the direction of increasing ξ, at a point above or below both curves, solutions cross $\xi + \eta = 1$ from above to below, while between the two, solutions cross in the other direction. Suppose then, that P_+ lies above $\xi + \eta = 1$; it will then lie to the right of $\xi = \gamma/(\gamma + 1)$, $\eta = 1/(\gamma + 1)$. Suppose also that $\mu < \gamma(\gamma + 1)$. Then a solution through P_μ crosses from above to below $\xi + \eta = 1$, and will certainly remain below until it meets $\gamma\eta - \xi = 0$ or $D = 0$. But since P_+ is above $\xi + \eta = 1$, the solution will - unless it goes to P_- - intersect $\gamma\eta - \xi = 0$ before $D = 0$. Hence it cannot go to P_+.

One further fact emerges from the above argument. From (7.6), the slope of the solution I_1 at P_1 is $-(1 + m)/2m$ and for the m given by (8.2),

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this is less than -1. Hence I_1 lies above $\xi + \eta = 1$ at P_1 and cannot cross it again without first crossing $\gamma\eta - \xi = 0$. Hence the solution I_+ through P_+ for the case just discussed, i.e., $\mu = \gamma/(\gamma + 1)$ must lie below I_1 at P_1, and hence must be a solution of our problem for the case of an exactly sonic shock.

Now consider a value of μ near $\gamma/(\gamma + 1)$; then m must be near $2\gamma/3(\gamma + 1)$. We set

$$\mu = \frac{\gamma}{\gamma + 1}\ (1 - \delta), \quad m = \frac{2\gamma}{3(\gamma + 1)}\ (1 + \varepsilon); \tag{8.3}$$

here, of course, $\delta > 0$. From (7.3), we have

$$\xi_+ = \frac{1}{6(\gamma + 1)}\left[4\gamma + 1 + (\gamma - 2)\varepsilon + (2\gamma - 1)\sqrt{1 - \frac{2(2\gamma^2 + 13\gamma + 2)\varepsilon}{(2\gamma - 1)^2} + \frac{(\gamma - 2)^2\varepsilon^2}{(2\gamma - 1)^4}}\right] \tag{8.4}$$

or

$$\xi_+ = \frac{\gamma}{\gamma + 1}\left[1 - \frac{3}{(2\gamma - 1)}\,\varepsilon - \frac{6(\gamma + 1)^2}{(2\gamma - 1)^3}\,\varepsilon^2 + \dots\right] \tag{8.5}$$

From the fact that P_+ lies below $\xi + \eta = 1$, we have $\varepsilon > 0$.

Next, we consider the solution I_+ through P_+. For $\xi^* = \xi_+$, $\eta^* = \gamma^{-1}\xi_+$ (7.11) reduces to

$$(3\gamma - 2m - 3\gamma\xi_+)\lambda^2 + \left[2(1 - \xi_+) + m\right]\lambda - \gamma^{-1}\left[(4\gamma - 1)\xi_+ - 2\gamma + 3 - (\gamma - 2)m\right] = 0 \tag{8.6}$$

Substituting (8.5) in (8.6), we find, to terms of order one in ε

$$\lambda = -\frac{\gamma + 3 + \sqrt{21\gamma^2 + 26\gamma + 54}}{5\gamma} + \frac{1}{25\gamma(2\gamma - 1)}\left[\frac{3(3\gamma + 4)(\gamma + 1) + 239\gamma^3 + 603\gamma^2 + 480\gamma + 126}{\sqrt{21\gamma^2 + 26\gamma + 54}}\right]\varepsilon + \dots \tag{8.7}$$

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Finally, substituting

$$\eta = \gamma^{-1}\xi^{-1} + \lambda(\xi - \xi^{+}) + \nu(\xi - \xi^{+})^2 + \dots \tag{8.8}$$

in (6.16), and using (8.5), (8.7), we obtain, to terms of order zero in ε

$$\nu = \frac{2(\gamma + 1)}{5\gamma^2} \; \frac{471 + 257\gamma + 81\gamma^2 - 3\gamma^3 + (72 + 25\gamma + 7\gamma^2)\sqrt{21\gamma^2 + 26\gamma + 54}}{30 + 79\gamma + 18\gamma^2 + 3(6\gamma + 5)\sqrt{21\gamma^2 + 26\gamma + 54}} \tag{8.9}$$

Setting

$$\varepsilon = a_1\delta + a_2\delta^2 + \dots , \tag{8.10}$$

(8.5), (8.7), (8.8), (8.9) are now sufficient to determine a_1, a_2; we give below some numerical results

$$\gamma = 5/3,\ a_1 = .309,\ a_2 = -.327;\ \gamma = 3,\ a_1 = .514,\ a_2 = -.772;\quad \gamma = 8,\ a_1 = .989,\ a_2 = -3.20 \tag{8.11}$$

Thus, we have, for m as a function of μ, near $\mu = \gamma/(\gamma + 1)$, a curve of the type shown in Fig. 5.

The shaded area in the figure has the following meaning: We observe that for μ near $\gamma/(\gamma + 1)$, the value of m given by (8.10) is not the only solution. For the ξ co-ordinate of P_0 is $(3\gamma - 3 - 2m)/(3\gamma - 1)$, and this is less than $\gamma/(\gamma + 1)$ for $m \sim 2\gamma/3(\gamma + 1)$, $\gamma > 0$. Hence P_+ is a sheaf-point, and the configuration of solutions about it is that shown in Fig. 6. Clearly for the value of m which this figure represents, we have as well as the solution I_+, corresponding to $\mu = \mu^*$, all solutions between I_+ and the integral curve $P_{\gamma/(\gamma + 1)}P_+$ as possible integrals from $\xi + \eta = 1$ to P_+,

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and thus all values of μ on the interval $\mu^* \leq \mu < \gamma/(\gamma + 1)$. Also as possible integrals P_+P_1, we have, as well as I_+, all curves below I_+, up to and including $P_+P_-P_1$ if $\xi_- \geq \xi_0$, all curves up to but not including $P_+P_0P_1$ if $\xi_- < \xi_0$. Thus, for $\mu \sim \gamma/(\gamma + 1)$, the pairs (μ, m) for which solutions exist are not only the points on the curves given by (8.10), shown in Fig. 5, but rather all points in the shaded area shown.

One fact, however, is to be observed with regard to the multiplicity of solutions. Of the curves through P_+, the integral I_+, and one of the family of positive slope are analytic there, as we have observed earlier; all others have infinite curvature. Thus we have certainly one analytic solution $P_{\mu^*}\ P_+\ P_1$, and possibly one other - the other will exist if the second analytic solution through P_+ lies between I_+ and $P_{\gamma/(\gamma+1)}P_+$ above $\gamma\eta - \xi = 0$, and between I_+ and $P_+P_-P_1$ $(P_+P_0P_1)$ below $\gamma\eta - \xi = 0$. Whether this second analytic solution exists or not can only be investigated numerically, and we shall not consider the question here. We may observe, however, that in the case of a precisely analogous question in the problem of a converging free surface (LA-210, p. 27) the answer was negative.

It will be seen from Fig. 6 that there are actually two solutions through $P_{\gamma(\gamma+1)}$, which go to P_1. Both correspond to exactly sonic shocks - the corresponding curve in the (y,t)-plane is an envelope of characteristics behind it. However, the one which lies always below $\gamma\eta - \xi = 0$, gives a curve in the (y,t)-plane which has at each point greater curvature than the characteristic through that point and so is in reality supersonic with reference to the material behind it.

Preserving the meaning given to μ^* in Fig. 6, i.e., the ξ co-ordinate of the intersection of I_+ with $\xi + \eta = 1$, we have now to consider how the function $m(\mu^*)$ whose asymptotic value for $\mu^* \longrightarrow \gamma/(\gamma + 1)$ is given by

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(8.3), (8.10) may behave as μ^* decreases to values $\ll \gamma/(\gamma + 1)$. We enumerate the possibilities which are critical.

(1) I_+ and I_1 through P_1 become coincident;

(2) I_+ and I_2 through P_2 become coincident;

(3) P_+ and P_- become coincident;

(4) P_+ and P_o become coincident.

The third and fourth possibilities are not critical in the same sense as the first two, and we shall consider first (4). If this happens for a particular $\mu^* = \mu_1$, $m = m_1$, without (1), (2), (3) having happened earlier, then for a slightly smaller value of μ^*, P_o will lie above P_+, and P_+ will be hyperbolic. Hence the multiplicity of solutions for a given m, corresponding to $\mu^* < \mu < \gamma/(\gamma + 1)$ will disappear at this point; beyond it - i.e., $m > m_1$, we will have only, for given m, the solution corresponding to $\mu = \mu^*$. This situation moreover, up to the point where (1), (2), or (3) occur, could only be altered by m reaching a value for which P_o is above $\xi + \eta = 1$. But, for this, m would have to be <u>less than</u> $2\gamma/3(\gamma + 1)$, since for all greater values, $\xi_o < \gamma/(\gamma + 1)$ is readily derived, and both numerical and geometric considerations indicate that $m(\mu^*)$ is monotone decreasing.

We turn next to the possibility (1). If this occurs, then I_+ no longer represents a solution of our problem, so we have to consider it. In the first place, we observe that for this to happen, m must attain a value for which I_1 at P_1 lies below $\xi + \eta = 1$, by virtue of the argument based on the equation (8.3) and following that equation in the above text. Since always $m \leq 2\gamma/(\sqrt{\gamma} + \sqrt{2})^2$, while the slope of I_1 at P_1 is $-(1 + m)/2m$, this can only

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happen for $2\gamma/(\sqrt{\gamma}+\sqrt{2})^2 > 1$ which is equivalent to $\gamma > 6 + 4\sqrt{2}$. Since we are not interested in values of γ of this order, we do not consider (1) further.

Now suppose (2) occurs. Then, insofar as the function $\eta(\xi)$ is concerned, we have at this point a solution of our problem corresponding to $\mu = 0$. For we can take as our integral the line $\xi = 0$ from $\xi = 0$, $\eta = 1$ to P_2, and the integral curve I_+ from P_2 to P_1. However, P_2 is a pole of s, so this does not satisfy (7.1). But consider a value of μ^* slightly larger than zero. Then I_+ will lie slightly above I_2, and we have in fact solutions for μ arbitrarily close to zero. Thus the region in the (μ, m)-plane for points in which solutions exist would look qualitatively as in Fig. 7. Here, if (4) has not occurred before (1), the point (μ_1, m_1) will not appear as shown in the figure, and the shaded portion of the plane will extend to the horizontal line through the intersection of $m = m(\mu^*)$ and $\mu = 0$. We have, of course, assumed implicitly that $m(\mu^*)$ remains single-valued and increases as μ^* decreases throughout the range $0 < \mu^* < \gamma/(\gamma+1)$. There is no way to establish this analytically, but it is borne out by rough geometric considerations as well as by numerical results. We do not, therefore, consider such possibilities as $dm/d\mu^* \to \infty$, etc.

Finally, suppose we have (3). Then for some value $\mu^* = \mu_0 > 0$, $m(\mu^*)$ takes on the value $m_0 = 2\gamma/(\sqrt{\gamma}+\sqrt{2})^2$ and the above analysis exhausts all cases in which I_+ is a solution. The continuation of the function $m(\mu^*)$ then obtains by passing to the integral I_- through P_-. For consider a value of m slightly less than m_0, and let μ_+, μ_- be the points of intersection of I_+, I_- with $\xi + \eta = 1$. Then it is reasonable to suppose $\mu_+ - \mu_- \sim \xi_+ - \xi_-$

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and from (7.3), $\xi_+ - \xi_- \sim (m_o - m)^{1/2}$. So at (μ_o, m_o) we have

$$m_o - m(\mu^*) \sim (\mu_o - \mu^*)^2 \qquad (8.12)$$

and thus $m(\mu^*)$ has a horizontal tangent at $\mu^* = \mu_o$, $m = m_o = 2\gamma/(\sqrt{\gamma} + \sqrt{2})^2$. The value μ_o must be found numerically in any given case.

Now let us return to a consideration of (4), and let us also consider the question of the coincidence of P_o, P_-. We have then $\xi_o = \xi_\pm$, or

$$\frac{3\gamma - 3 - 2m}{3\gamma - 1} = \frac{2\gamma + (\gamma - 2)m \pm\sqrt{[2\gamma + (\gamma - 2)m]^2 - 8\gamma^2 m}}{4\gamma}, \qquad (8.13)$$

or

$$2\gamma(3\gamma - 5) - (3\gamma^2 + \gamma + 2)m = \pm(3\gamma - 1)\sqrt{[2\gamma + (\gamma - 2)m]^2 - 8\gamma^2 m} \qquad (8.14)$$

In particular, suppose all three points P_o, P_-, P_+ are coincident. Then

$$m = 2\gamma/(\sqrt{\gamma} + \sqrt{2})^2 = 2\gamma(3\gamma - 5)/(3\gamma^2 + \gamma + 2)$$

and this yields

$$3(2\gamma)^{3/2} - 10(2\gamma)^{1/2} - 8 = 0 \qquad (8.15)$$

This equation has one real root, namely,

$$\gamma = 2.289730 \qquad (8.16)$$

Now suppose γ is less than this root. Then the left side of (8.14) is negative for $m = 2\gamma/(\sqrt{\gamma} + \sqrt{2})^2$ and increases to $2\gamma(3\gamma - 5)$ as m decreases to zero. The right side (with the negative sign) decreases from zero to $-2\gamma(3\gamma - 1)$. Hence for $\gamma > 1$, there will be a value of m for which the two sides are equal, that is for which the points P_o, P_- coincide. On the other hand, suppose γ exceeds the root of (8.15). Then for $m = 2\gamma/(\sqrt{\gamma} + \sqrt{2})^2$,

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the right side of (8.14) is positive and P_o lies to the right of P_+. Since we have already seen that it lies to the left for $m = 2\gamma/3(\gamma + 1)$, there is an intermediate value for which the two coincide.

In either of the above cases, or in the exceptional case when P_o, P_-, P_+ all coincide ((8.15) holds), the value of m where P_o coincides with P_-, P_+, is obtained by solving (8.14). Squaring both sides and collecting terms we have

$$6\gamma(\gamma - 1) - 3(2\gamma^2 - \gamma + 1)m - (3\gamma^2 - 3\gamma + 2)m^2 = 0 \qquad (8.17)$$

This equation has only one positive root for $\gamma > 1$, so P_o coincides with P_- or P_+ for exactly one value of m on the interval $0 < m \leq 2\gamma/(\sqrt{\gamma} + \sqrt{2})^2$. The coincidence is with P_-, both, or P_+ according as γ is less than, equal to, or greater than the root γ_o of (8.15). We also observe that for $\gamma = \gamma_o$, $m \sim m_o = 2\gamma_o/(\sqrt{\gamma_o} + \sqrt{2})^2$, ξ_- decreases as m decreases, ξ_+ increases with $(m_o - m)^{1/2}$, ξ_o with $(m_o - m)$. So for $\gamma = \gamma_o$, P_o lies always between P_-, P_+.

We are now in position to continue our analysis of the dependence of m on μ. Consider first the solution I_- through P_- for values of m near $2\gamma/(\sqrt{\gamma} + \sqrt{2})^2$. If we do not have for some $m < m_o$, the situation pictured in Fig. 7, and already discussed, then for $m \sim m_o$, I_- intersects $\xi + \eta = 1$ at a point $\xi = \mu^*$, $\eta = 1 - \mu^*$, $\mu^* > 0$. The point P_- will be a sheaf-point or a hyperbolic point, moreover, according as P_o lies to its right or its left, that is, according as $\gamma \geq \gamma_o$, or $\gamma < \gamma_o$. Hence in the second case, for the value of m in question, there is only the solution I_-, while in the first the solutions in the neighborhood of P_- look as in Fig. 4. Not only

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does the curve I_- intersect $\xi + \eta = 1$, but all integral curves entering P_- from above and lying above the solution I_2 at P_2 do also, giving values of μ on the interval $0 < \mu < \mu^*$. Moreover, not only can we take I_- as the integral from P_- to P_1, but also any integral curve between I_- and the broken curve $P_-P_oP_1$.

It remains to be considered what may happen on the left-hand branch of the function $\mu^*(m)$ as m decreases to values $\ll 2\gamma/(\sqrt{\gamma} + \sqrt{2})^2$. The possibilities are as follows:

(1) I_- becomes coincident with I_1 through P_1;

(2) $\mu^* \longrightarrow 0$, $m \longrightarrow 0$;

(3) I_- becomes coincident with I_2 through P_2;

(4) P_- becomes a spiral point.

We dispose of the first at once. By an argument previously given it certainly cannot happen for $\gamma \leqq 6 + 4\sqrt{2}$, and we are not concerned with larger γ.

With regard to (3), (4) both, we first observe that neither is possible so long as the minimum point of $N = 0$ lies to the left of $\xi = 0$, that is as long as

$$3 - \left[\frac{12\gamma - 2(\gamma - 3)m}{2\gamma - 1}\right]^{1/2} < 0$$

by (7.12), and thus as long as

$$3(2\gamma - 3) + 2(\gamma - 3)m < 0$$

This holds for all m if $\gamma \leqq 3/2$, so for such γ we have definitely (2). For $\xi \longrightarrow 0$, the solutions of our equation apart from I_2 through P_2 are asymptotically

$$\eta^{3\gamma - 2m}\, \xi^{3\gamma} = \text{const.}$$

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so for $m \sim 0$, $\eta >> 0$,

$$\eta \xi = \text{const.}$$

Thus the desired solution is asymptotically

$$\xi \eta = \mu$$

for $m \longrightarrow 0$. Since $\xi_- \sim \eta_- \sim m$, we have, for $\gamma \leqq 3/2$, and the curve $m(\mu^*)$ has a vertical tangent at $m = 0$.

Now consider $\gamma > 3/2$, $m \sim 0$. To terms of order 1 in m, we have $\xi_- = m/2$, $\eta = m/2\gamma$, while the η-co-ordinate of P_2 is $m/3$, and the slope of I_2 is $-(2\gamma - 3)m/9\gamma$. Hence, for $\xi = \xi_-$, the η-co-ordinate of the corresponding point on I_2 is $m/3$, to terms of order one in m, while $\eta_- = m/2\gamma$. Hence, for some value of $m > 0$, either (3) or (4) obtains.

Our analysis of the interdependence of m, μ is now as complete as it can be made without resort to numerical methods. In particular, the question of whether (3) or (4) holds for $\gamma > 3/2$ can only be determined numerically. If (3) holds, then $\mu^*(m)$ approaches zero as m approaches some value greater than zero, while if (4) holds there is a point $\mu^* > 0$, $m > 0$ on the curve $\mu = \mu^*(m)$ beyond which it cannot be defined, and the region in the (m, μ)-plane for whose points solutions exist will have a qualitatively different appearance. In Figs. 8 we show in a qualitative way the nature of this region. In each of the various cases pictured in Figs. 8a to 8d the complete curve $\mu = \mu^*(m)$ is drawn assuming (3). If (4) holds instead, the left-hand end of the shaded portion has the appearance shown in Fig. 8e. Note that in all cases there is an interval of values of μ for which a unique m exists. This interval of course becomes a point for $\gamma = \gamma_0$. Note also, that except on this interval each point of the region pictured represents not one, but a one-parameter family of solutions, by virtue of the multiplicity of solutions

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from $P_{\pm}$ to P_1 in any given case. Finally, it is to be recalled that for each point on the curve $\mu = \mu^*(m)$ it is possible to find an analytic solution $P_\mu P_{\pm} P_1$, namely $I_{\pm}$. It is possible also that the shaded regions contain arcs along which this is possible (Cf pp. 35), but in general, solutions corresponding to points in the shaded regions lead to solutions of the original problem in which a sound wave converges to the origin behind the shock, arriving simultaneously with it. It seems reasonable, therefore, in the absence of other criteria, to prefer the solutions corresponding to points on the curve $\mu = \mu^*(m)$.

Figs. 8 are, of course, to be taken as giving nothing more than a qualitative picture of the solution; quantitative results can only be obtained by numerical analysis. In § 10, we shall give some numerical data for various γ.

In conclusion, we note that in addition to the solutions we have already taken account of, there are others as well, in which more than one converging shock wave is present. For, first consider the solution I_+ from P_+ to P_1. For $\mu \sim \gamma/(\gamma + 1)$, we have from (8.7) that the slope of I_+ at P_+ is less than -1. Hence the point $(\xi_2 \eta_2)$ into which a point $(\xi_1 \eta_1)$ near P_+ on I_+, below $\gamma\eta - \xi = 0$, is transformed by (6.18), (6.19) lies to the right of I_+, and of course above $\gamma\eta - \xi = 0$ and below $\xi + \eta = 1$. Hence it lies on an integral curve which can be taken into P_+. That is, from the point $\xi_1 \eta_1$, we can pass by shock to ξ_2, η_2 then back to P_+ and thence to P_1. Or this procedure can be repeated as often as one likes.

Next, consider the behavior of the point $(\xi_2 \eta_2)$ as $(\xi_1 \eta_1)$ moves away from P_+ on I_+. As (ξ_1, η_1) approaches P_1, $\xi_2 \to 0$, $\eta_2 \to 1$. So for some

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$(\xi_1\eta_1)$, $(\xi_2\eta_2)$ lies on I_+ above P_+, and we can pass to $(\xi_2\eta_2)$ by shock, and then again along I_+ to $(\xi_1\eta_1)$. This also can be repeated as often as one likes.

Now, suppose $\mu^* << \gamma/(\gamma + 1)$, but still greater than μ_0, and consider the point $(\xi_2\eta_2)$ as $(\xi_1\eta_1)$ varies from P_+ to P_1. Again $(\xi_2\eta_2)$ moves from P_+ to $\xi_2 = 0$, $\eta_2 = 1$, and hence, if I_- intersects $\xi + \eta = 1$, $(\xi_2\eta_2)$ lies on I_- for some $(\xi_1\eta_1)$. In this case, we can pass by shock from the point $(\xi_1\eta_1)$ to $(\xi_2\eta_2)$ on I_-, and thence to P_- and P_1.

Finally, consider the solution I_- from P_- to P_1, in the case that P_- is a sheaf-point. As (ξ_1, η_1) moves along I_-, the point (ξ_2, η_2) into which it is transformed by the shock conditions, must ultimately cross all solutions between I_- and I_2 through P_2, above $\gamma\eta - \xi = 0$, and since all of these go to P_- without crossing that line, we can pass by shock to any one of them. This process also can be repeated as often as one likes.

The circumstances under which it is not clear from the above that solutions involving more than one convergent shock exist are these: (1) the case pictured in Fig. 7 with $\mu << \gamma/(\gamma + 1)$; (2) the case of a solution through P_+ if $\mu \to 0$ on the left hand branch of the curves shown in Figs. 9, for a value of $m >> 2\gamma/3(\gamma + 1)$; (3) the case of a solution through P_-, when P_- is a hyperbolic point. Whether or not there are pairs (μ, m) falling under one or another of these cases and for which no further shock is possible, is a question which can only be investigated numerically.

9. <u>The Solution for the Outward Motion.</u> From the discussion in § 6 of the boundary conditions (b) for $t = 0$, it is clear that the solution is continued beyond P_1 to the region $t > 0$ (in the ordinary sense), simply by taking the

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analytic continuation of the solution $P_{\pm} P_1$ for $\xi > 1$. This solution continues to a point $(\xi_1 \eta_1)$ for which the $(\xi_2 \eta_2)$ given by (6.18), (6.19) lies on a curve entering P_o, or on the curve through P_∞, $\xi = (3\gamma - 3 - 2m)/(3\gamma - 1)$, $\eta = \infty$. Since the latter is not a singular point, there is only one solution through it.

We shall now show that the latter is the only possibility. For, suppose P_o is above $\gamma\eta - \xi = 0$. Then it lies either to the right of P_+ or to the left of P_-, and is thus a sheaf-point, according to § 7. The configuration of integral curves about it is therefore either that shown in Fig. 9a, or that shown in Fig. 9b, the arrows indicating the direction of decreasing s. Since s must increase to $+\infty$ at $x = 0$, it follows that none of these solutions is acceptable; on all of them $s \rightarrow -\infty$, as $(\xi, \eta) \rightarrow P_o$.

Next consider the case that P_o lies below $\gamma\eta - \xi = 0$. Then the configuration of integral curves about it is that shown in Fig. 4. Of the two curves through it, I_o is the only one along which s increases to $+\infty$ as $(\xi, \eta) \rightarrow P_o$. But the point $(\xi_2 \eta_2)$ must lie above $\gamma\eta - \xi = 0$ and I_o cannot cross $\gamma\eta - \xi = 0$ at either P_- or P_+. So in this case also, a solution ending at P_o is impossible.

The only possibility, therefore, is the solution ending at

$$\xi = \frac{3(\gamma - 1) - 2m}{3\gamma - 1}, \quad \eta = \infty$$

Note that $\xi = \left[3(\gamma - 1) - 2m\right]/(3\gamma - 1)$ is the vertical asymptote of $D = 0$, and that by simple geometrical considerations, the integral curve in question remains between this asymptote and $D = 0$ until it enters the uppermost of the points P_o, P_+, which it must do. To find the point on it corresponding to the shock, one has in general to integrate both (6.6) and (6.9) through the point

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P_1, and then taking as (ξ_1, η_1) a point on the continuation of the (ξ, η)-curve through P_1, form the (ξ_2, η_2) of this point by (6.18), (6.19) and find the $(\xi_2 \eta_2)$ which lies on the curve which is shown above to correspond to the high-pressure side of the shock. Only in the case $\mu = (\gamma - 1)/(\gamma + 1)$, $C = 0$, when g disappears from (6.19), is a simpler alternative available. In this case we can form the (ξ_1, η_1) of all (ξ_2, η_2) on the curve through P_∞, and then seek the point of intersection of the resulting (ξ_1, η_1)-curve with the solution from P_1.

10. <u>Numerical Results</u>. For the determination of the function $\mu = \mu^*(m)$, we have as a starting point, for any value of γ, the point $\mu = \gamma/(\gamma + 1)$, $m = 2\gamma/3(\gamma + 1)$, and by use of equations (8.3) - (8.10), we can find the slope and curvature at this point, as illustrated in equation (8.11). A natural next step is to determine $\mu_0 = \mu^*$ for $m = 2\gamma/(\sqrt{\gamma} + \sqrt{2})^2$, where we know $dm/d\mu = 0$. This we have done for $\gamma = 5/3$, $\gamma = 3$, $\gamma = 8$, finding, respectively, $\mu_0 = .3118$, $\mu_0 = .2530$, $\mu_0 = .065$. This provides us with five data to which to fit the desired curve, namely, the two points $\mu = \gamma/(\gamma + 1)$, $m = 2\gamma/3(\gamma + 1)$; $\mu = 2\gamma/(\sqrt{\gamma} + \sqrt{2})^2$, $m = m_0$; the slope and curvature at the first; and the slope at the second. Approximations so determined have been found quite accurate, except near $\mu = 0$; thus, for $\gamma = 5/3$, the approximation gives $m = .453$, for $\mu = (\gamma - 1)/(\gamma + 1) = .25$, while actual integration with $m = .453$, gives $\mu = .252$. Similarly, for $\gamma = 3$, the approximation gives $m = .572$ for $\mu = .50$, while actual integration with $m = .572$ gives $\mu = .497$.

For the sake of completeness, in one case, $\gamma = 3$, the curve $\mu = \mu^*(m)$

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has been investigated in the neighborhood of $\mu = 0$. It was found that for $m = .433$, I_- and I_2 become coincident, so $\mu = 0$, $m = .433$ is the left-hand endpoint on this curve. In addition, the point $m = .5$, $\mu = .038$ was found.

On the basis of these various numerical results, we have provided, in Figs. 10a, 10b, 10c, approximate graphs showing the regions in the (μ, m)-plane for points in which solutions exist, for the three values $\gamma = 5/3$, $\gamma = 3$, $\gamma = 8$, respectively, and omitting the lower left-hand portions for $\gamma = 5/3$, $\gamma = 8$. These graphs are believed to be accurate to within two per cent.

In the case $\gamma = 3$, $m = .572$ $(n = .636)$, $\mu = .497$, a complete numerical integration of the problem has been carried out. Fig. 11 shows in the (ξ, η)-plane the curve $P_\mu P_+ P_1 P_{S_1}$, where P_{S_1} corresponds to the low-pressure side of the reflected shock, and the curve $P_{S_2} P_\infty$, P_{S_2} corresponding to the high-pressure side of that shock. The critical curves $N = 0$, $D = 0$, $\gamma n - \xi = 0$ are also shown in the figure. In Fig. 12, the function $s = -\log w$ in its dependence on ξ, is given. From these two curves, all physically pertinent functions can be determined. In particular, for the inward motion, $x = awt^n$, $y = af(w)t^n$, while the position of the shock in the two frames of reference is $x_S = aw_S t^n$, $y_S = aw_S t^n$, $w_S = \left[(1 - \mu)\mu^\gamma\right]^{1/2(1 + m)}$. So $x/y_S = w/w_S$, $y/y_S = f(w)/w_S$, and $f(w)$ gives us y/y_S as a function of x/x_S. This function is shown in Fig. 13. Moreover, all physical quantities are functions of f and thus of y/y_S multiplied by appropriate scaling factors and powers of t, so that by a suitable choice of units (depending on time), all such quantities can be expressed as functions of y/y_S alone. Thus, if U is shock velocity, p_S pressure at the shock front, both at time t, then u/U, p/p_S are functions of y/y_S alone; similarly ρ/ρ_0 is a function of y/y_S

[redacted]

[redacted]

alone. These functions, ρ/ρ_o, u/U, p/p_S are graphed in Figs. 14, 15, 16 respectively. In order to have comparable graphs for the outward motion, it is now convenient to imagine the incident shock mirror-reflected in the y-axis, thus preserving the meanings of x/y_S, y/y_S, u/U, p/p_S. Adopting this procedure, we have shown, in Figs. 17, 18, 19, 20, x/y_S, ρ/ρ_o, u/U, p/p_S as functions of y/y_S. Thus for example, to read from Fig. 20 the value of the pressure p at a point y_o and time t_o after collapse one finds the values of y_S and p_S at the time t_o before collapse, and finds p/p_S at y_o/y_S. A table of values of ξ, n, S, x/y_S, y/y_S, ρ/ρ_o, u/U, p/p_S, over both epochs of the problem is given in Table 1.

Explanation of Tables and Graphs. As pointed out in the abstract, a value of μ (that is, the value of v on the incident shock), leads in general to a range of values of n, of which one corresponds to an analytic solution. Thus, if $m = 1 - n/n_o$ there will be a function $m(\mu)$ defined on the interval $o < \mu < \gamma/(\gamma + 1)$ corresponding to pairs (m,μ) for which analytic solutions exist, and other points in the plane corresponding to pairs (m,μ) for which non-analytic solutions exist. The possible (qualitative) nature of these functions is shown in Figs. 7, 8a to 8d, and in each figure, the shaded portions represent the regions corresponding to non-analytic solutions. The value of γ_o is 2.2897^+. Fig. 8e shows a variant of the left-hand end of the figure which is possible rather than that shown in Figs. 8b to 8d. No instances of this have been found, however, and in addition, no instances of Fig. 7 have been found. If the latter occurs at all, it is only for very large γ - that is $\gamma > 8$.

For $\gamma = 5/3$, $\gamma = 3$, $\gamma = 8$, the function $m(\mu)$ discussed in the preceding paragraph has been determined to within about 2 per cent, except for $\mu \sim 0$

[redacted]

in the cases $\gamma = 5/3$, $\gamma = 8$, and is shown in Figs. 10a, 10b, 10c.

For $\gamma = 3$, $\mu = .497$, we have $m(\mu) = .572$ $(n = .636)$, and for this case, a complete numerical integration of the problem has been carried out. If y_S, U, p_S represent respectively the position, velocity, and pressure of the incident shock at time t, then by virtue of the assumption of similarity, x/x_S, ρ/ρ_o, u/U, p/p_S, where x is the Lagrangian radius, ρ is density, ρ_o is normal density, u is velocity, p is pressure, are all functions of y/y_S. These functions are shown in Figs. 13, 14, 15, 16.

When the incident shock reaches the center, it is reflected and moves out through the material again. To see this epoch of the motion on the same scale as the incoming, we imagine the incident shock mirror-reflected in the y-axis, so that x/y_S, ρ/ρ_o, u/U, p/p_S, remain functions of y/y_S. These are shown in Figs. 17, 18, 19, 20. That is, for example, to find p at a time t_o after collapse and positions y_o, one finds y_S and p_S at the time t_o before collapse, and then determines p/p_S at y_o/y_S from Fig. 20.

The last five columns of Table 1 give the results described in the two preceding paragraphs, in tabular form.

[redacted]

ξ	η	S	x/x_S	y/y_S	ρ/ρ_o	u/U	p/p_S
.497	.503	- .8856	1.000	1.000	2.012	.5030	1.000
.56	.3771	- .9974	1.118	1.062	2.085	.4674	.987
.62	.2821	-1.1082	1.250	1.133	2.165	.4304	.966
.68	.2050	-1.2307	1.412	1.226	2.244	.3925	.935
.74	.1419	-1.3682	1.620	1.352	2.325	.3516	.887
.80	.0905	-1.5330	1.911	1.535	2.408	.3071	.817
.86	.0495	-1.7481	2.370	1.837	2.496	.2572	.712
.92	.0190	-2.0784	3.297	2.467	2.595	.1973	.548
.95	.0083	-2.3564	4.353	3.200	2.650	.1600	.425
.97	.0037	-2.6626	5.384	3.911	2.689	.1173	.259
1.99	.0004	-3.3160	11.361	8.152	2.734	.0815	.147
1.00 *	0 *	- ∞ **	∞*	∞ *	2.759 *	0 *	0 *
1.01	.00045	-3.3188	11.394	8.075	2.782	-.0807	.164
1.0301	.005	-2.5691	5.385	3.762	2.846	-.1132	.412
1.0582	.030	-2.0314	3.145	2.140	3.001	-.1245	.867
1.0644	.060	-1.8361	2.587	1.746	3.057	-.1124	1.183
1.0635	.075	-1.7757	2.435	1.637	3.096	-.1039	1.315
1.0588	.10	-1.7011	2.258	1.511	3.148	-.0888	1.515
1.0454	.14	-1.6165	2.077	1.384	3.235	-.0628	1.802
1.0267	.18	-1.5591	1.961	1.303	3.318	-.0347	2.071
{1.010 / .820} **	{.21 / .40} **	{-1.5273 / -1.5213} **	1.900 **	1.262 **	{3.377 / 4.160} **	{-.0126 / .2272} **	{2.268 / 4.320} **
.83126	.45455	-1.4247	1.724	1.165	3.900	.1966	3.964
.84293	.55556	-1.2936	1.513	1.044	3.608	.1640	3.663
.85201	.71429	-1.1466	1.306	.922	3.337	.1364	3.431
.85936	1.00000	- .9645	1.088	.788	3.060	.1108	3.258
.86537	1.66667	- .7047	.839	.630	2.729	.0848	3.106
.87054	5.00000	- .1710	.492	.396	2.197	.0513	2.99
.87163	10.00000	+ .1466	.358	.300	1.943	.0385	2.99
.87289	∞	∞	0	0	0	0	2.99

* These values correspond to $t = 0$ (or $x = \infty$).
** These values correspond to the reflected shock.

Table I

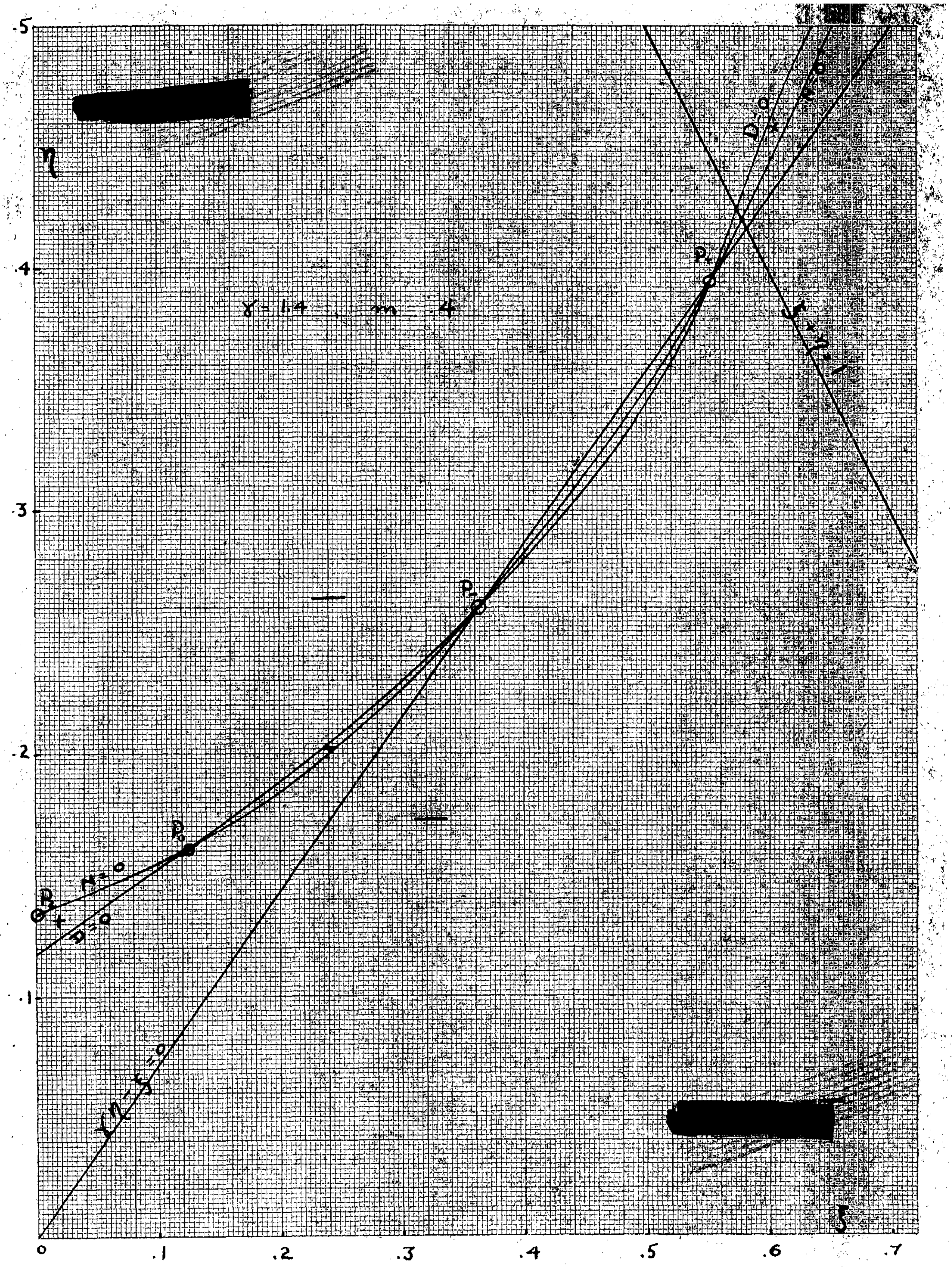

Fig. 1a

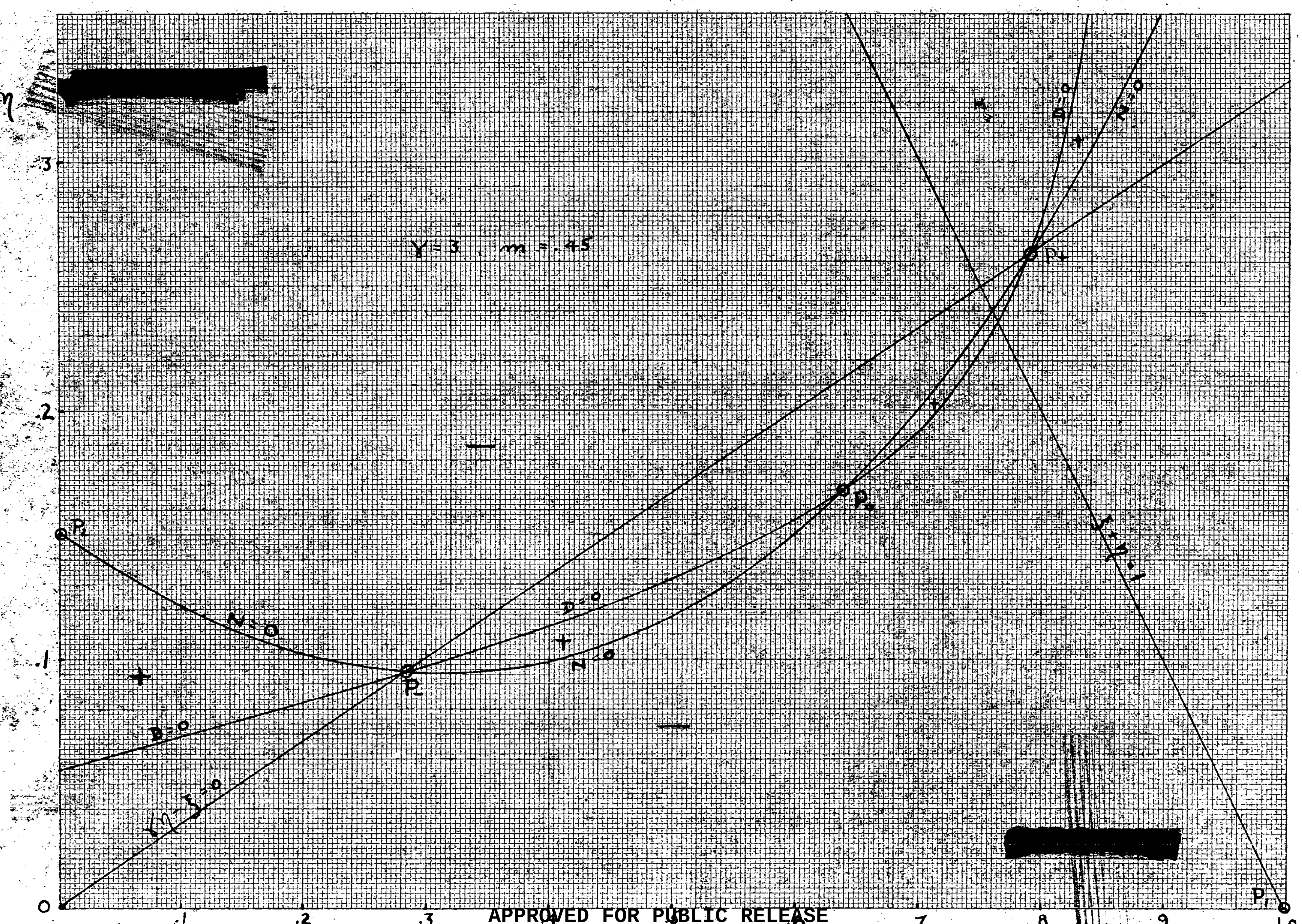

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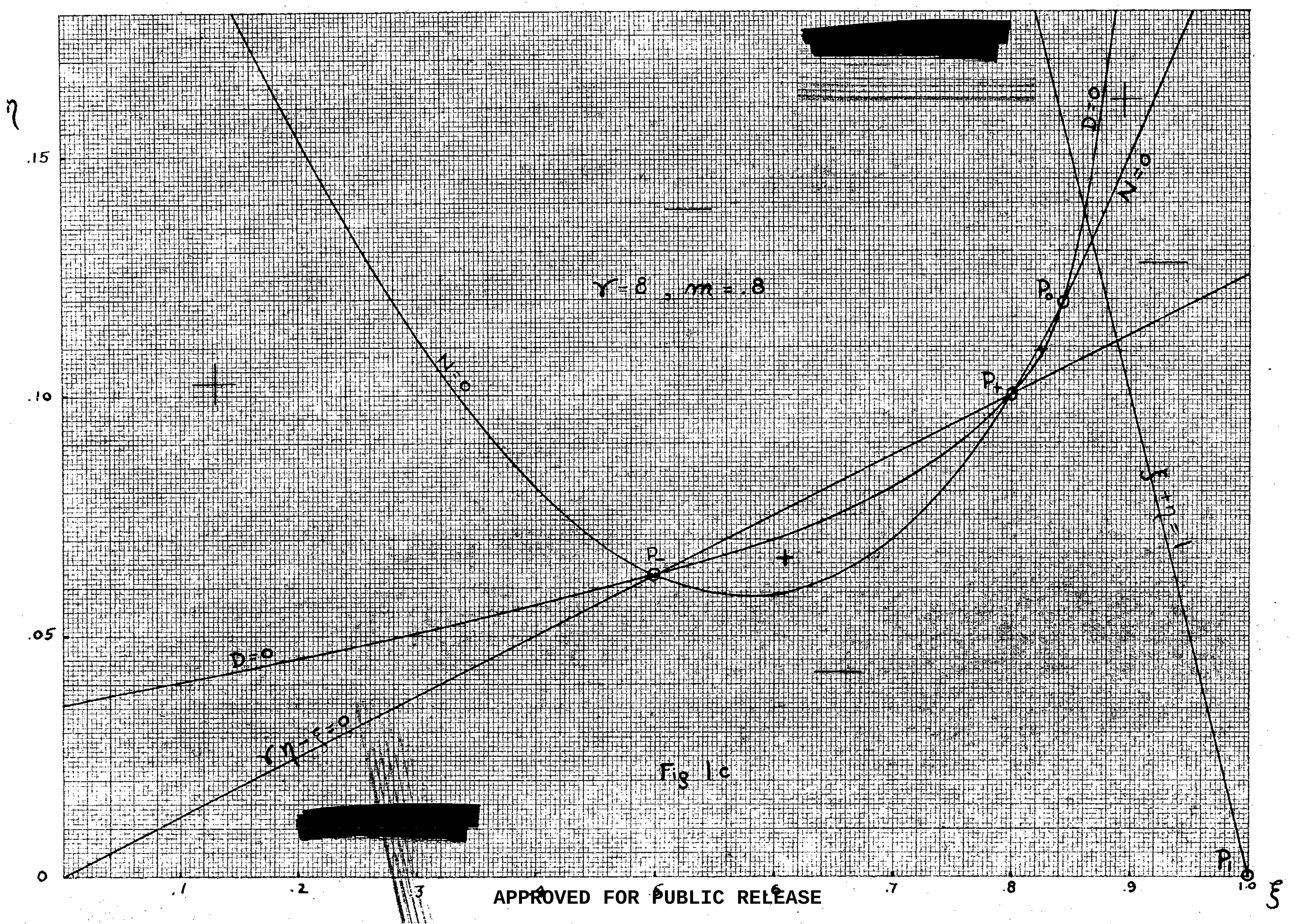
Fig. 1b

[redacted]



[redacted]

Fig 1c

[redacted]

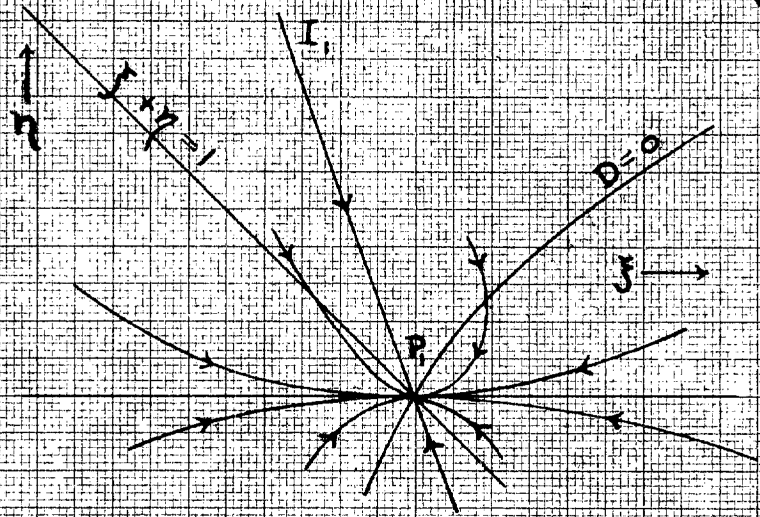


Fig 2

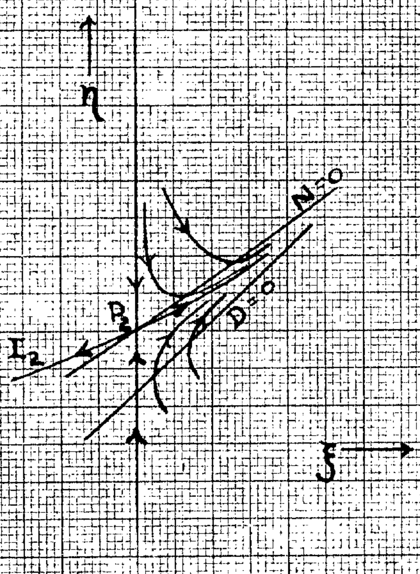


Fig 3a

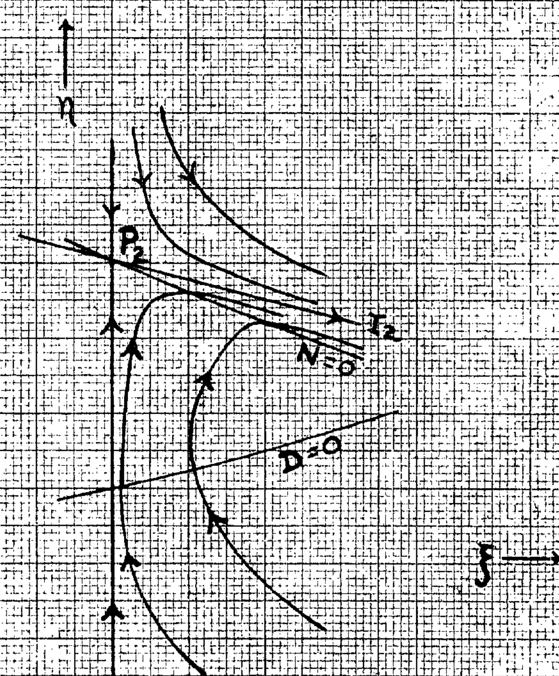
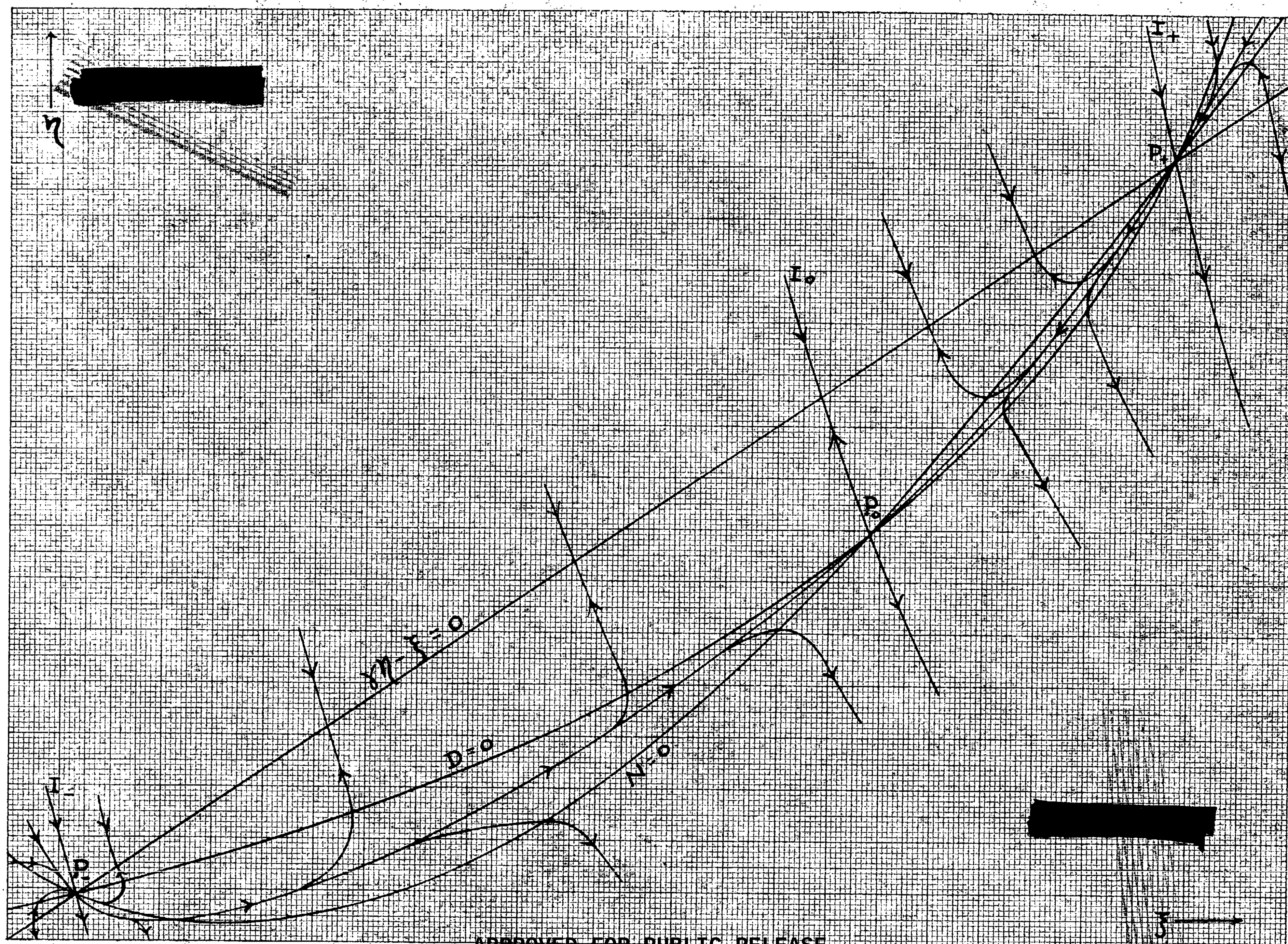


Fig 3b

[redacted]

[redacted]



[redacted]

Fig 4

[redacted]

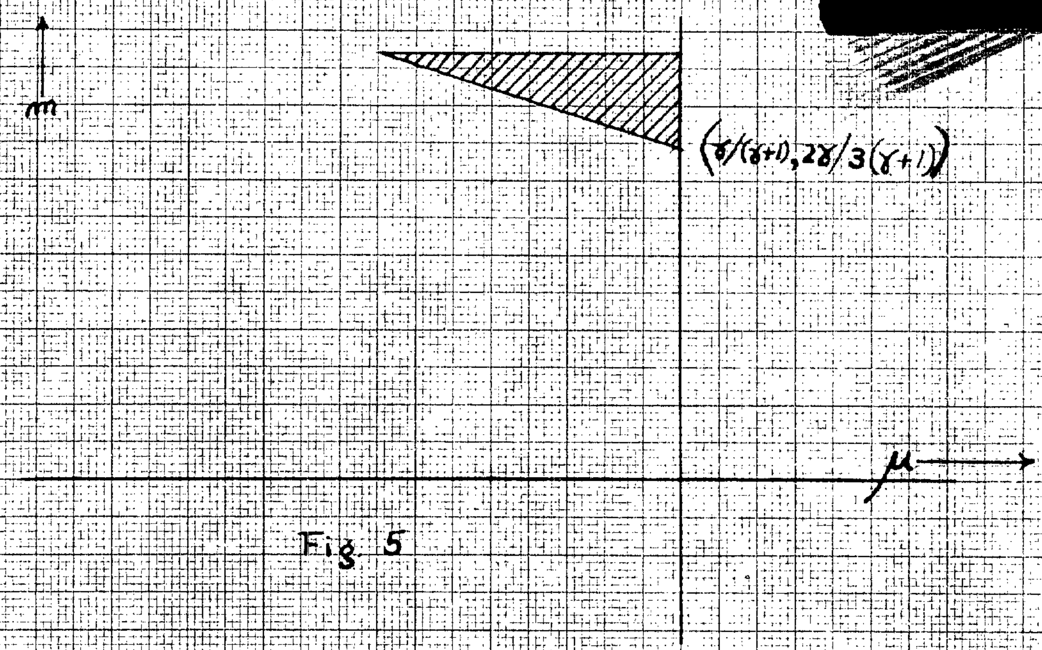


Fig. 5

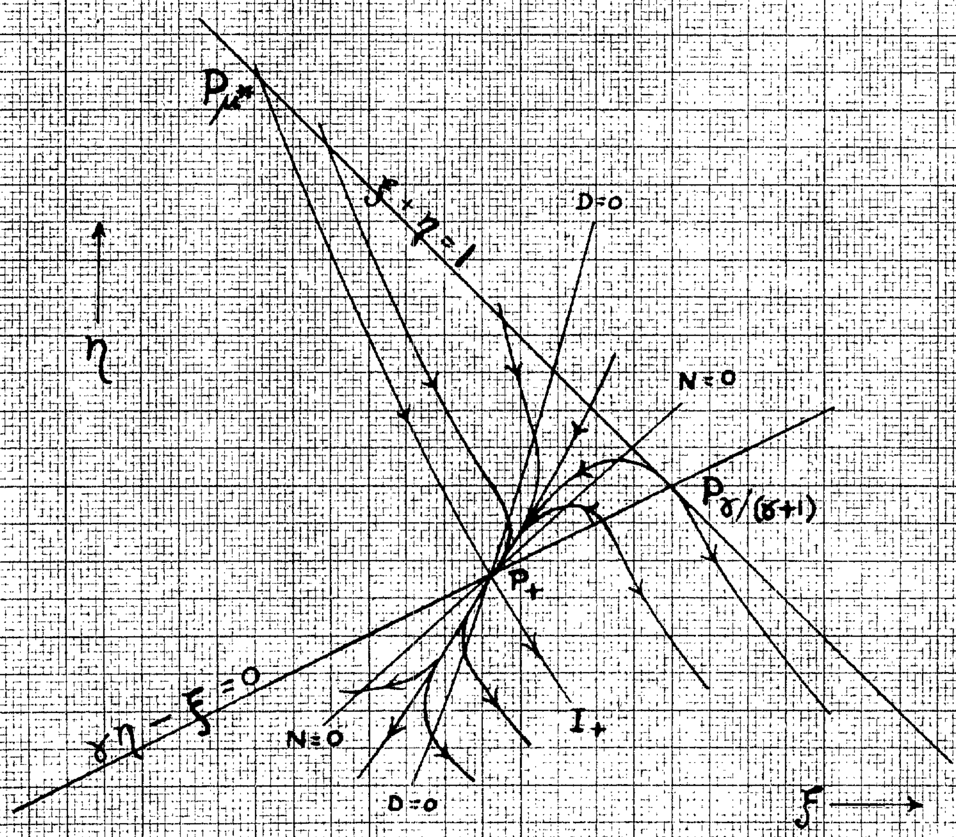


Fig. 6

[redacted]

[redacted]

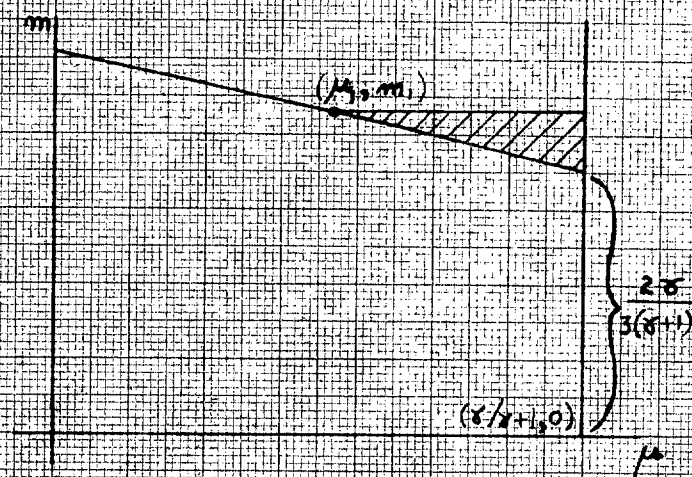


Fig 7

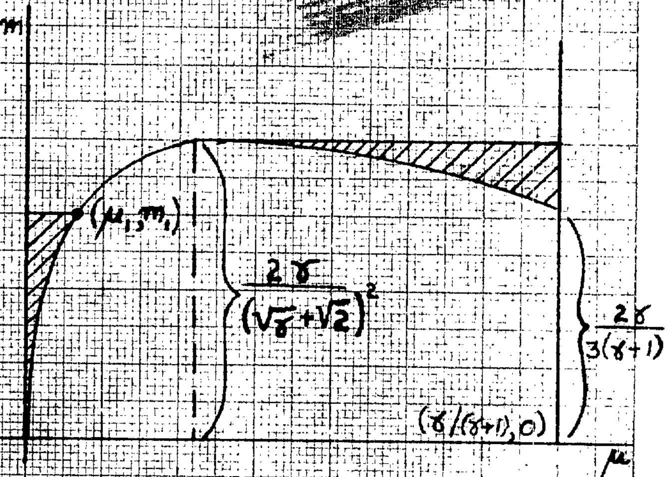


Fig. 8a $(\gamma \leq 3/2)$

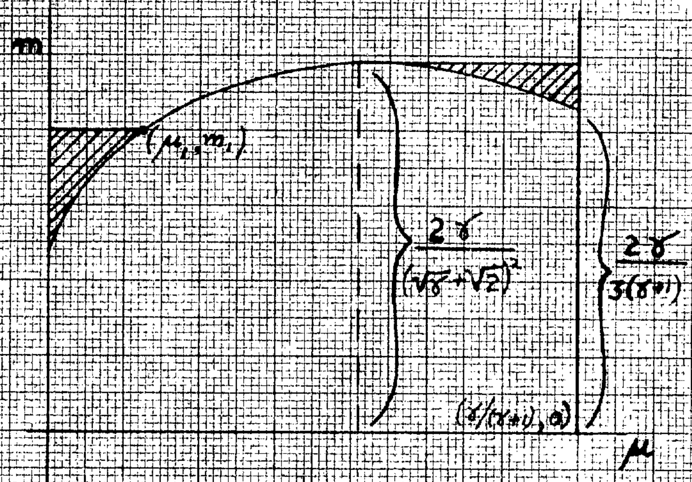


Fig 8b $(3/2 < \gamma < \gamma_0)$

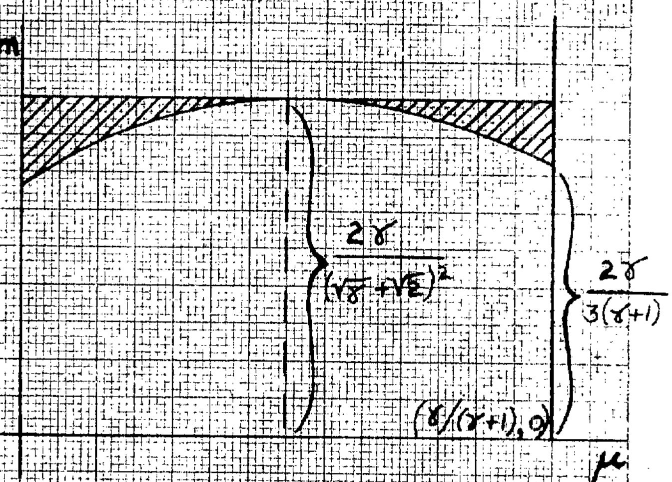


Fig 8c $(\gamma = \gamma_0)$

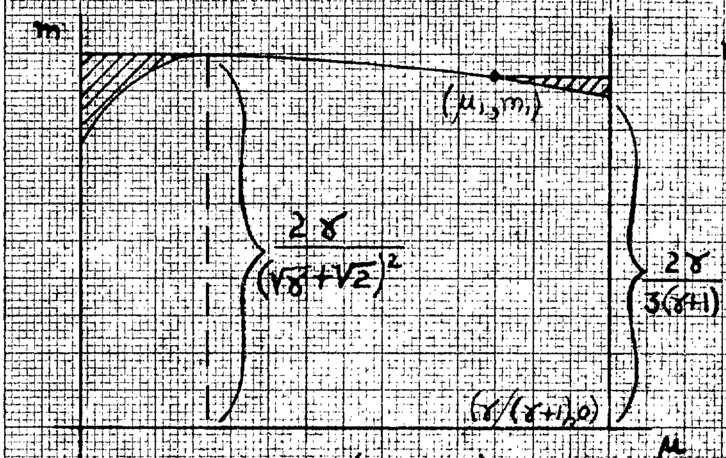


Fig. 8d $(\gamma > \gamma_0)$

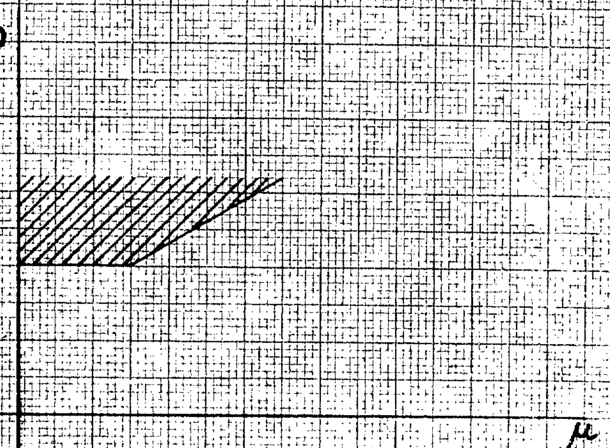


Fig 8e

[redacted]

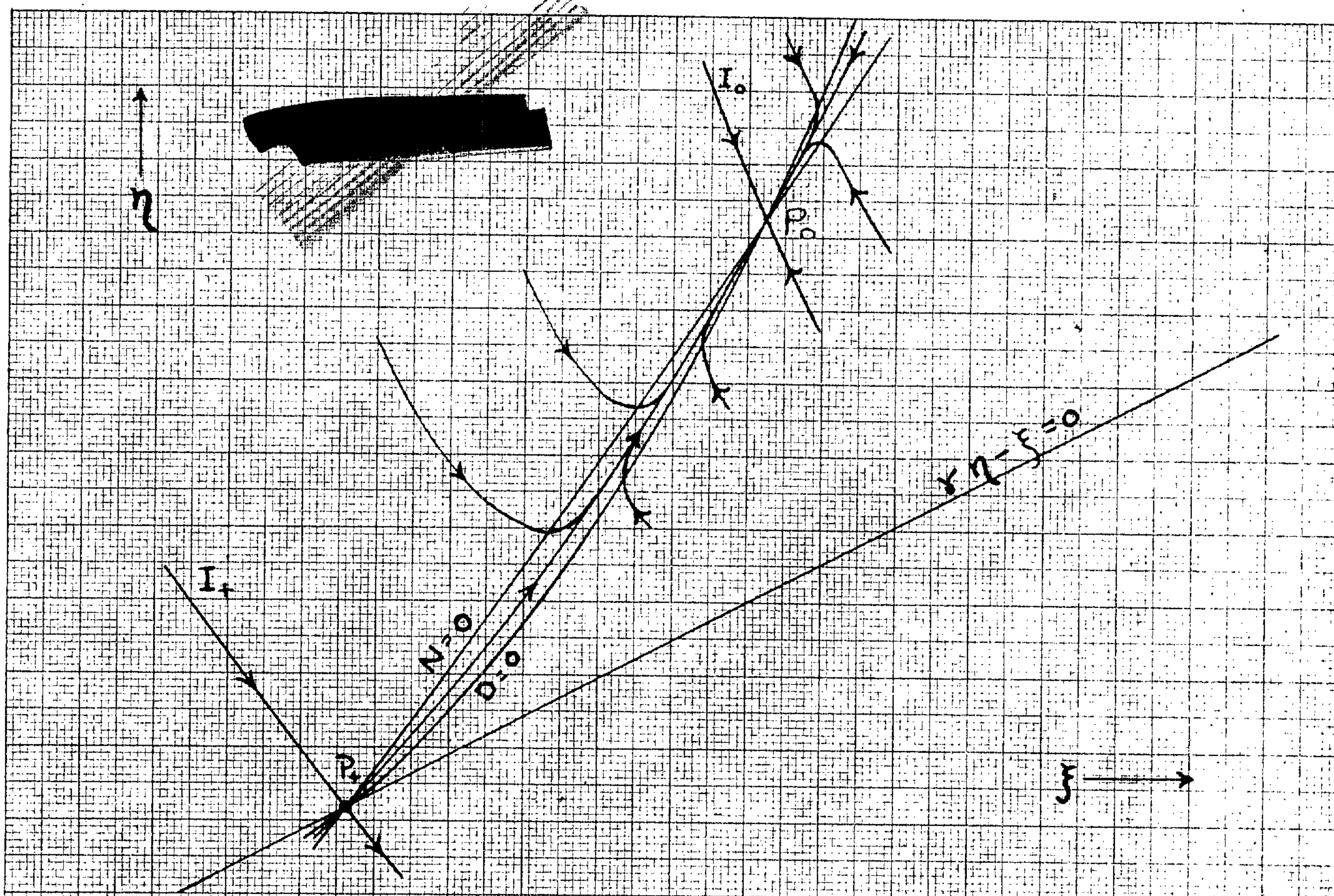

[redacted]
η
I_0
P_0
I_+
N=0
D=0
P_+
$\gamma\eta-\xi=0$
ξ

Fig 9a

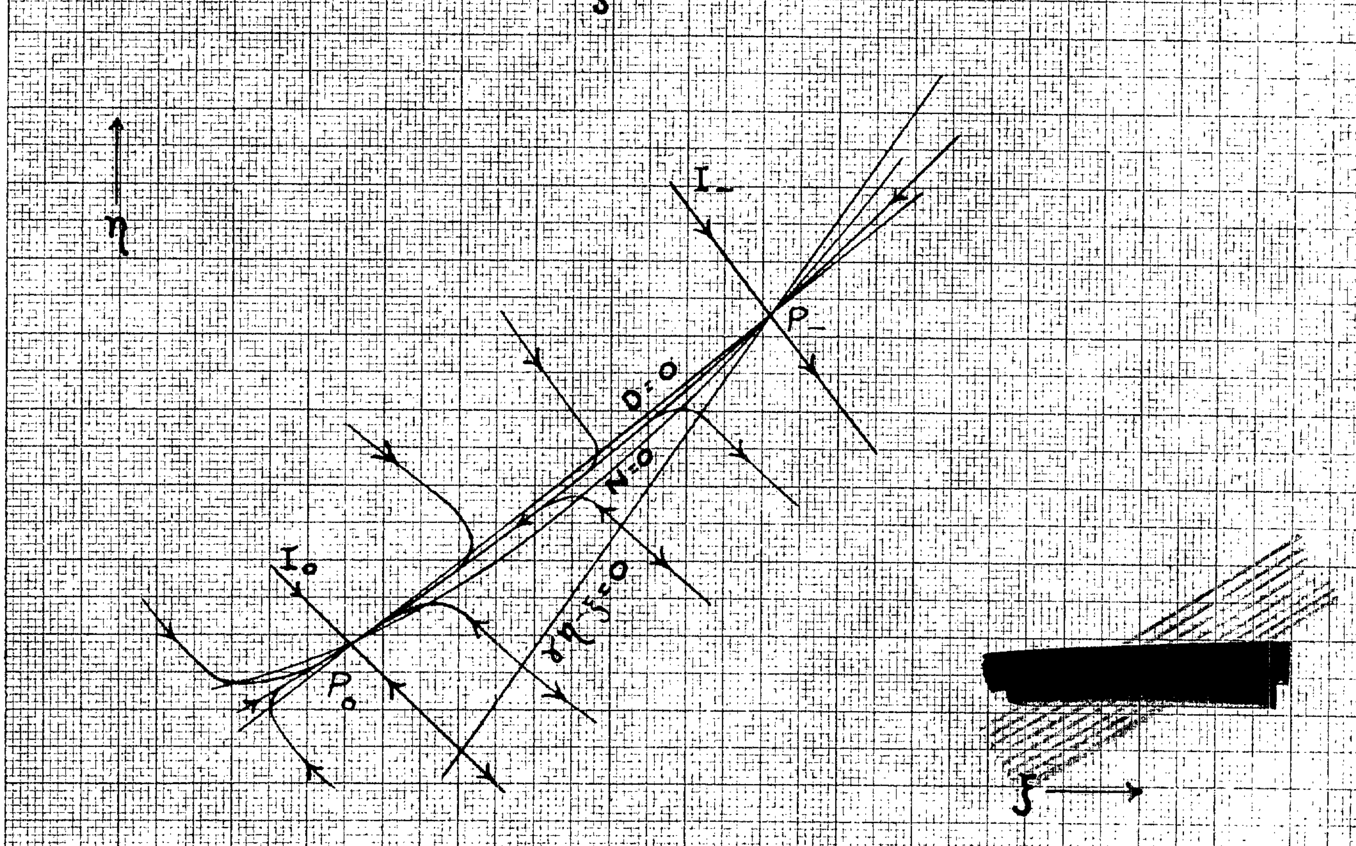

η
I_-
P_-
D=0
N=0
$\gamma\eta-\xi=0$
I_0
P_0
[redacted]
ξ

Fig 9b

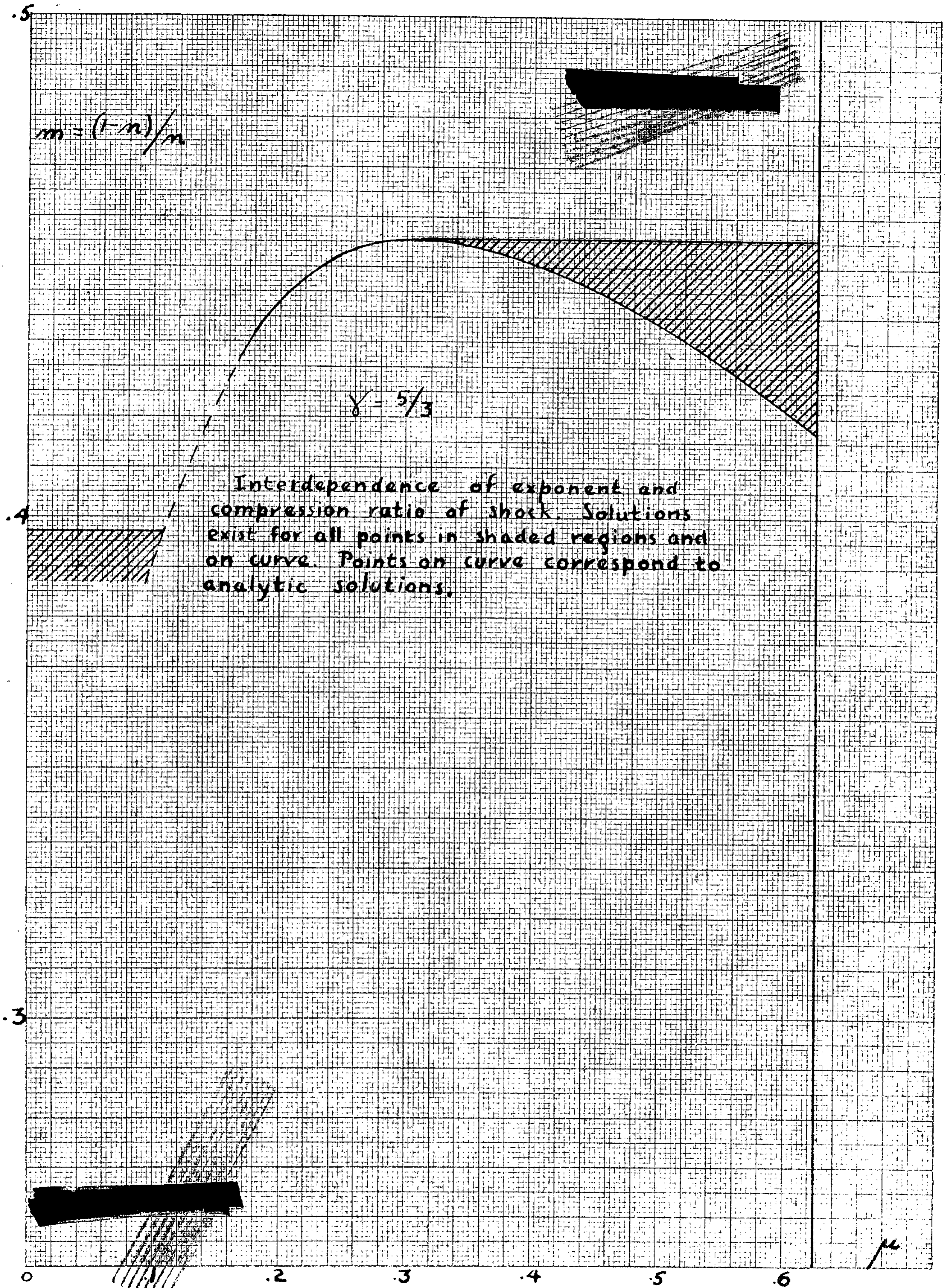

.5
m = (1-n)/n
[redacted]
γ = 5/3
Interdependence of exponent and compression ratio of shock. Solutions exist for all points in shaded regions and on curve. Points on curve correspond to analytic solutions.
.4
.3
[redacted]
0
.2
.3
.4
.5
.6
μ

Fig 19

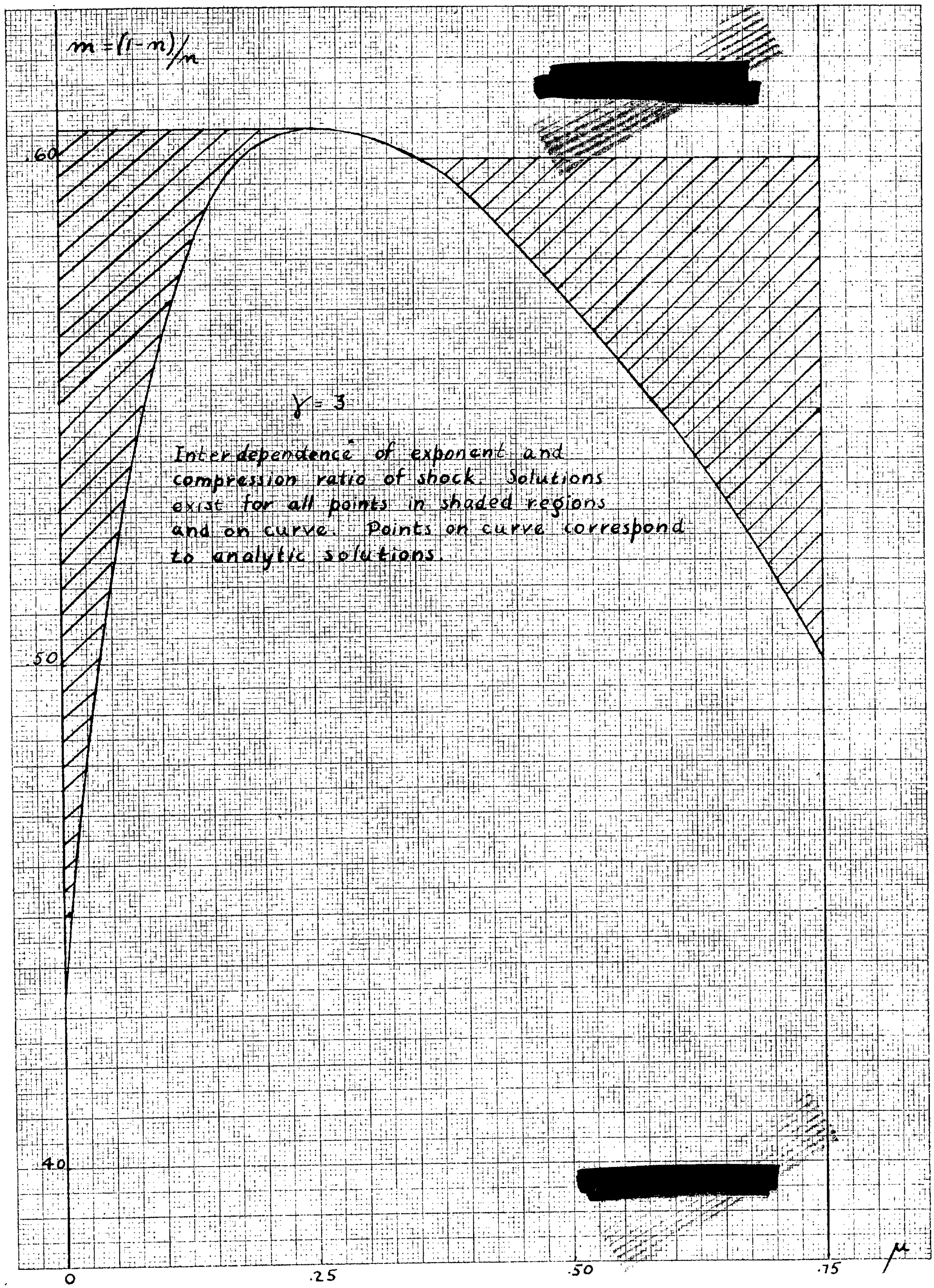
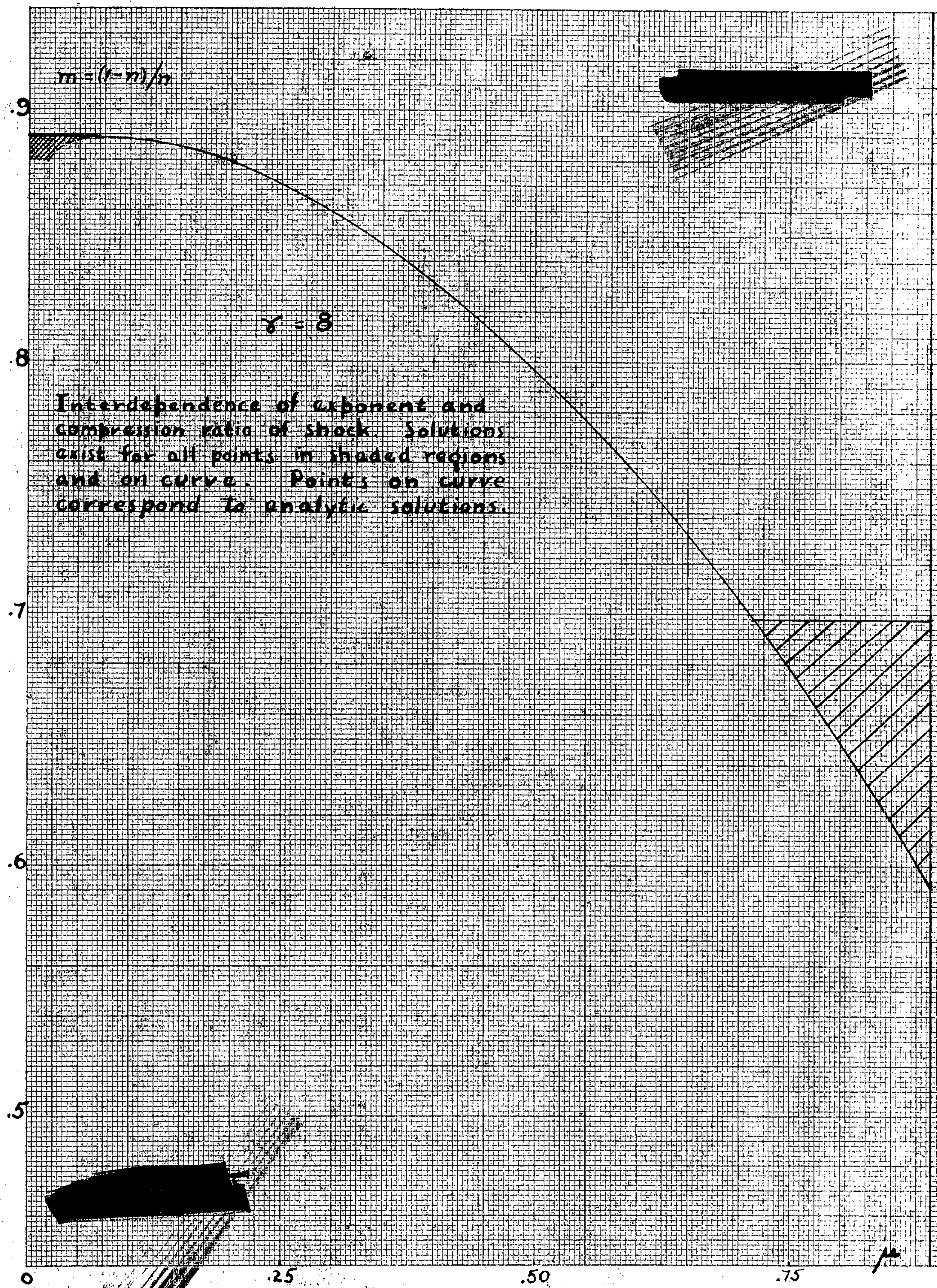

m = (1-n)/n
[redacted]
.60
.50
.40
γ = 3
Interdependence of exponent and compression ratio of shock. Solutions exist for all points in shaded regions and on curve. Points on curve correspond to analytic solutions.
[redacted]
0
.25
.50
.75
μ

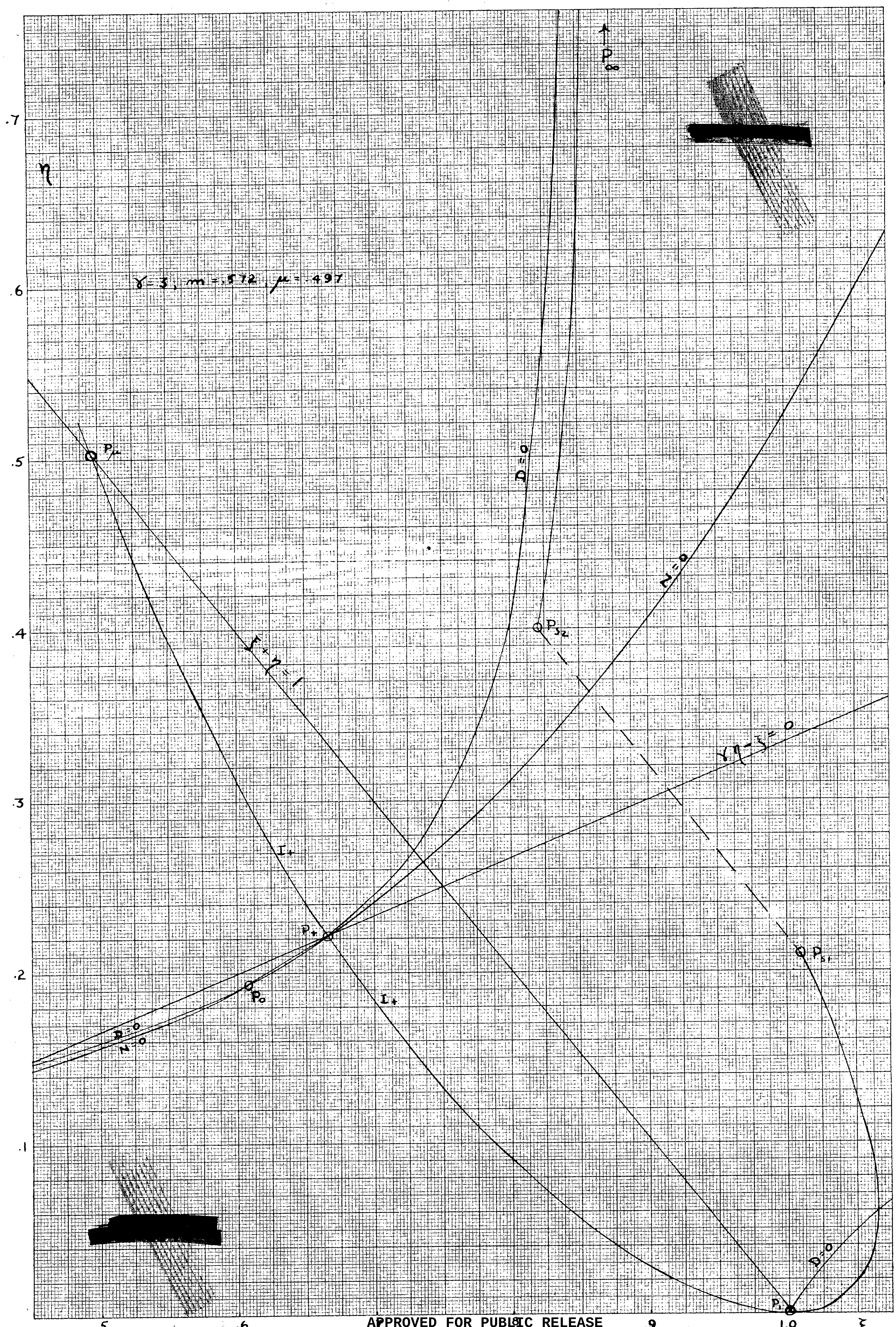
Fig 10 b



m = (1-n)/n
γ = 8
Interdependence of exponent and compression ratio of shock. Solutions exist for all points in shaded regions and on curve. Points on curve correspond to analytic solutions.
.9
.8
.7
.6
.5
0
.25
.50
.75
μ

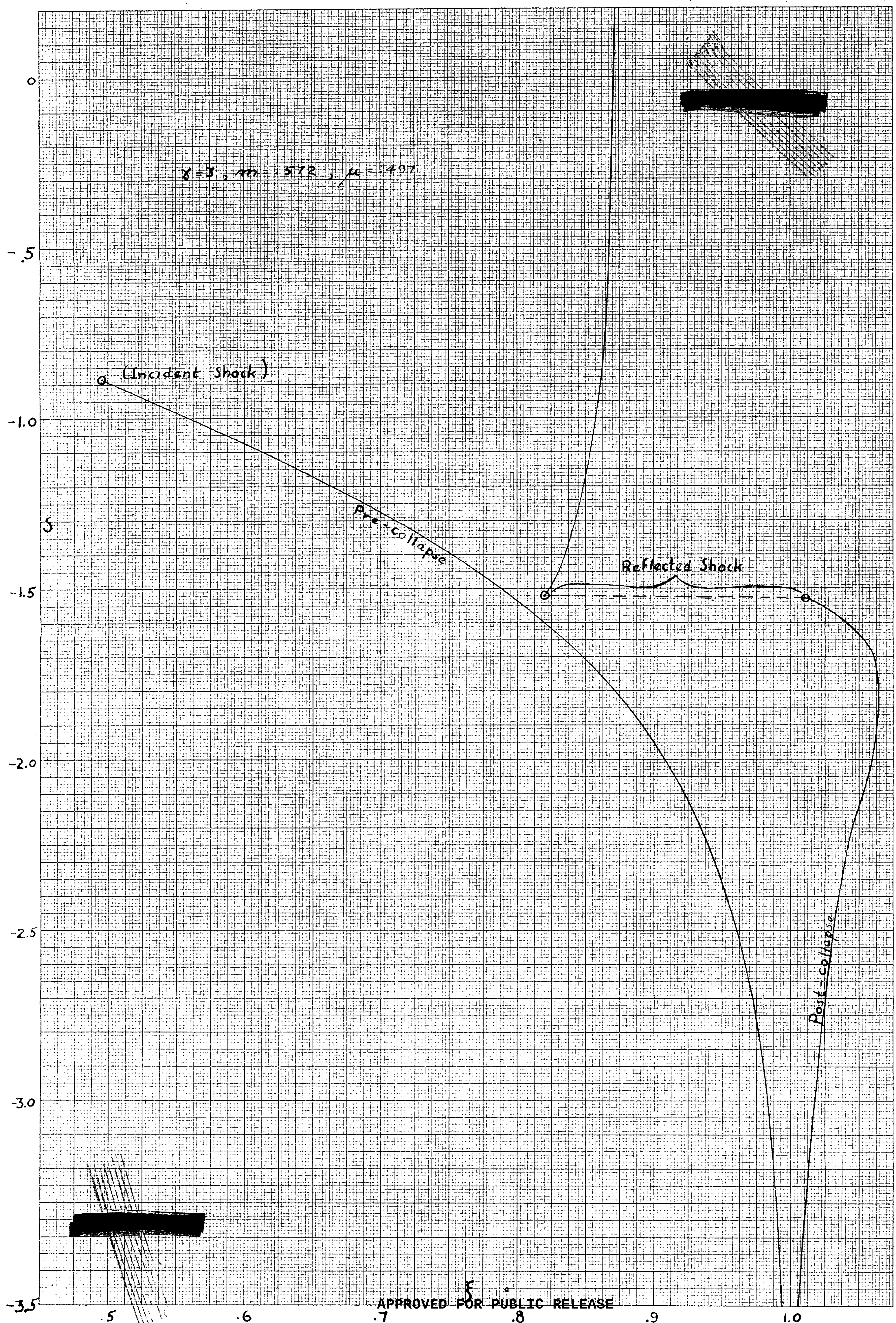
Fig. 10c

[redacted]



[redacted]

Fig. 11


[redacted]
γ = 3 , m = .572 , μ = .497
(Incident Shock)
Pre-collapse
Reflected Shock
Post-collapse
[redacted]
0
-.5
-1.0
-1.5
-2.0
-2.5
-3.0
-3.5
S
.5
.6
.7
.8
.9
1.0
ξ

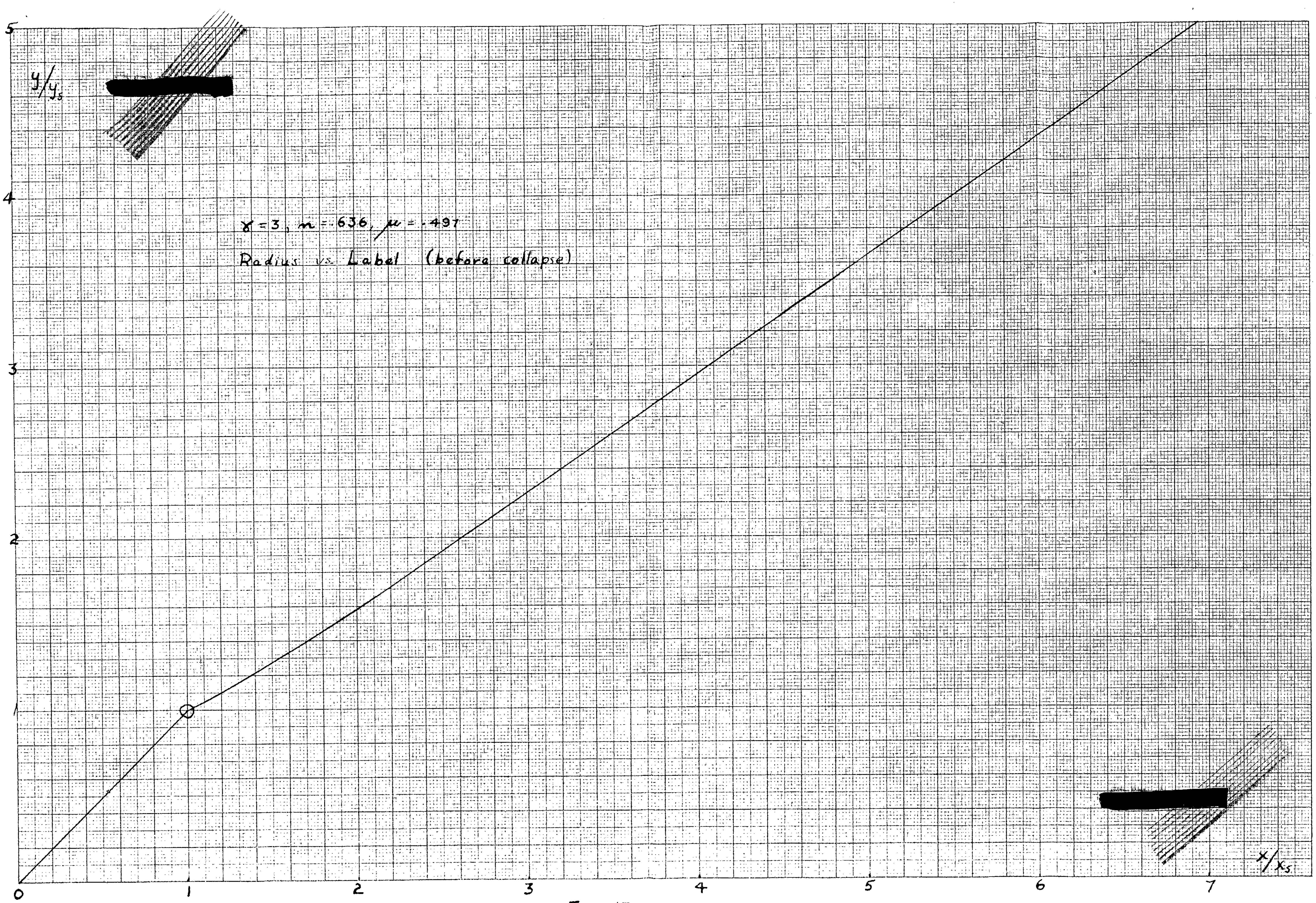

y/y_s
[redacted]
γ = 3, n = .636, μ = .497
Radius vs. Label (before collapse)
[redacted]
x/x_s
0
1
2
3
4
5
6
7

Fig 13

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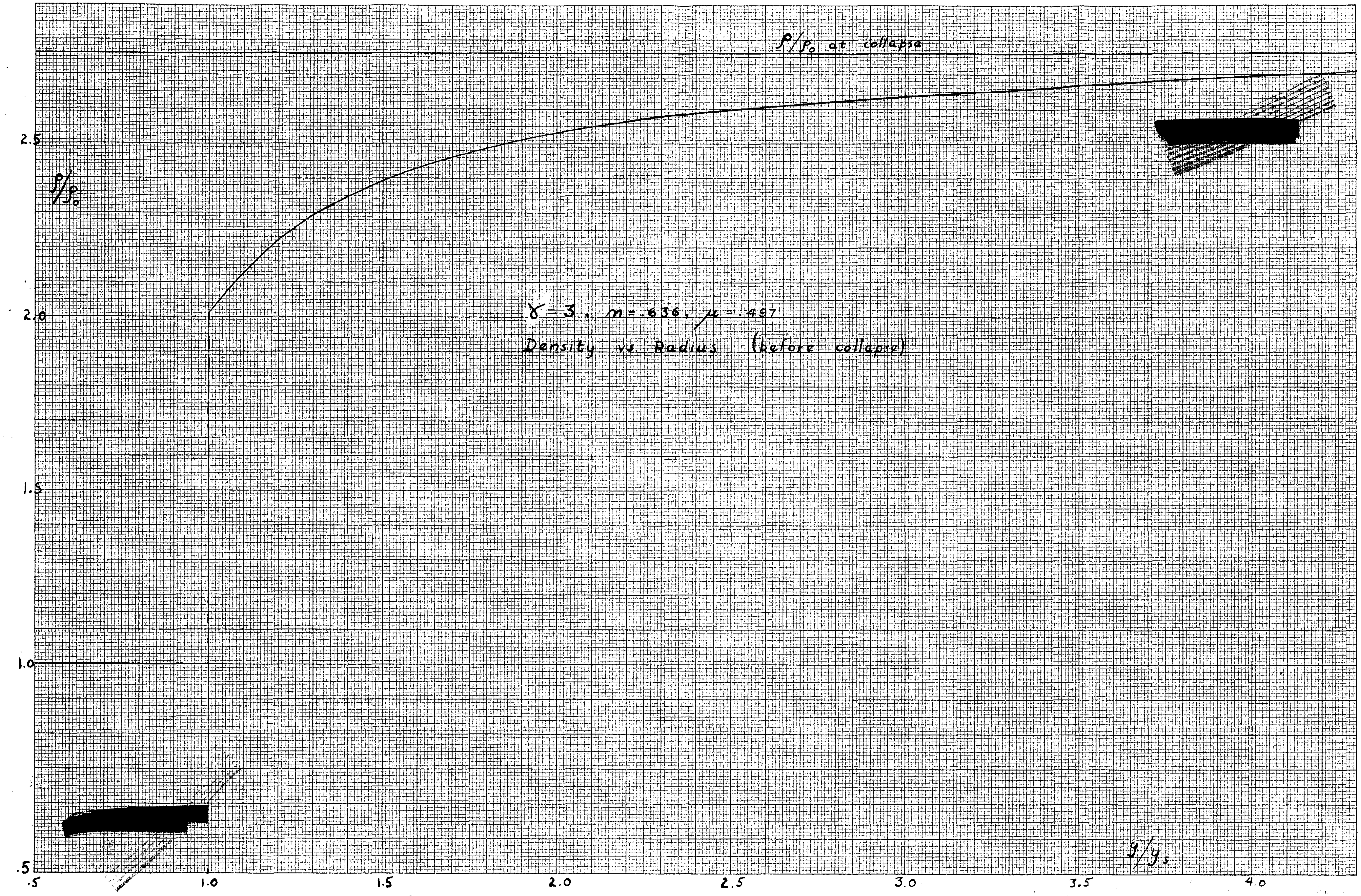

Fig 14

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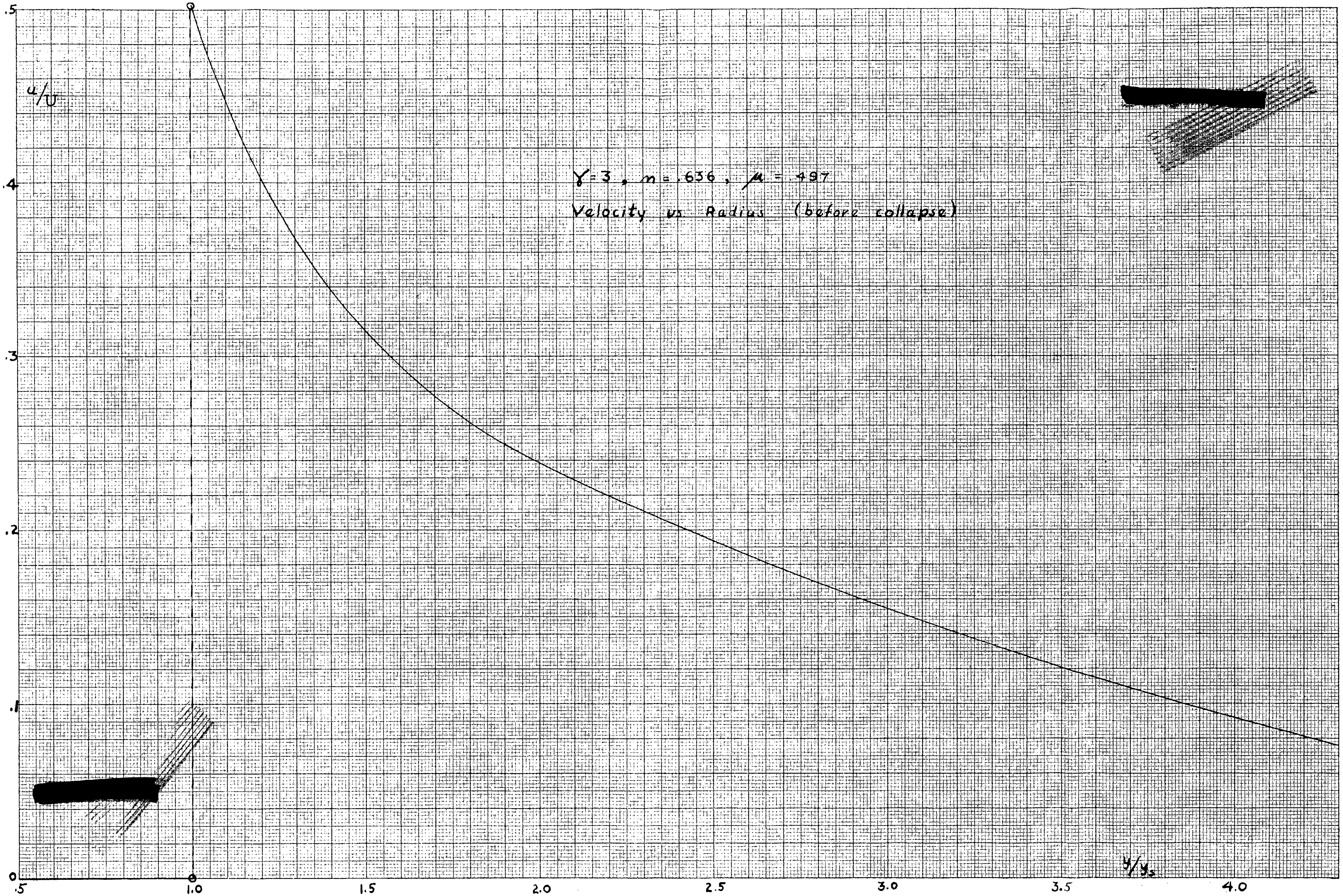

Fig 15

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APPROVED FOR PUBLIC RELEASE

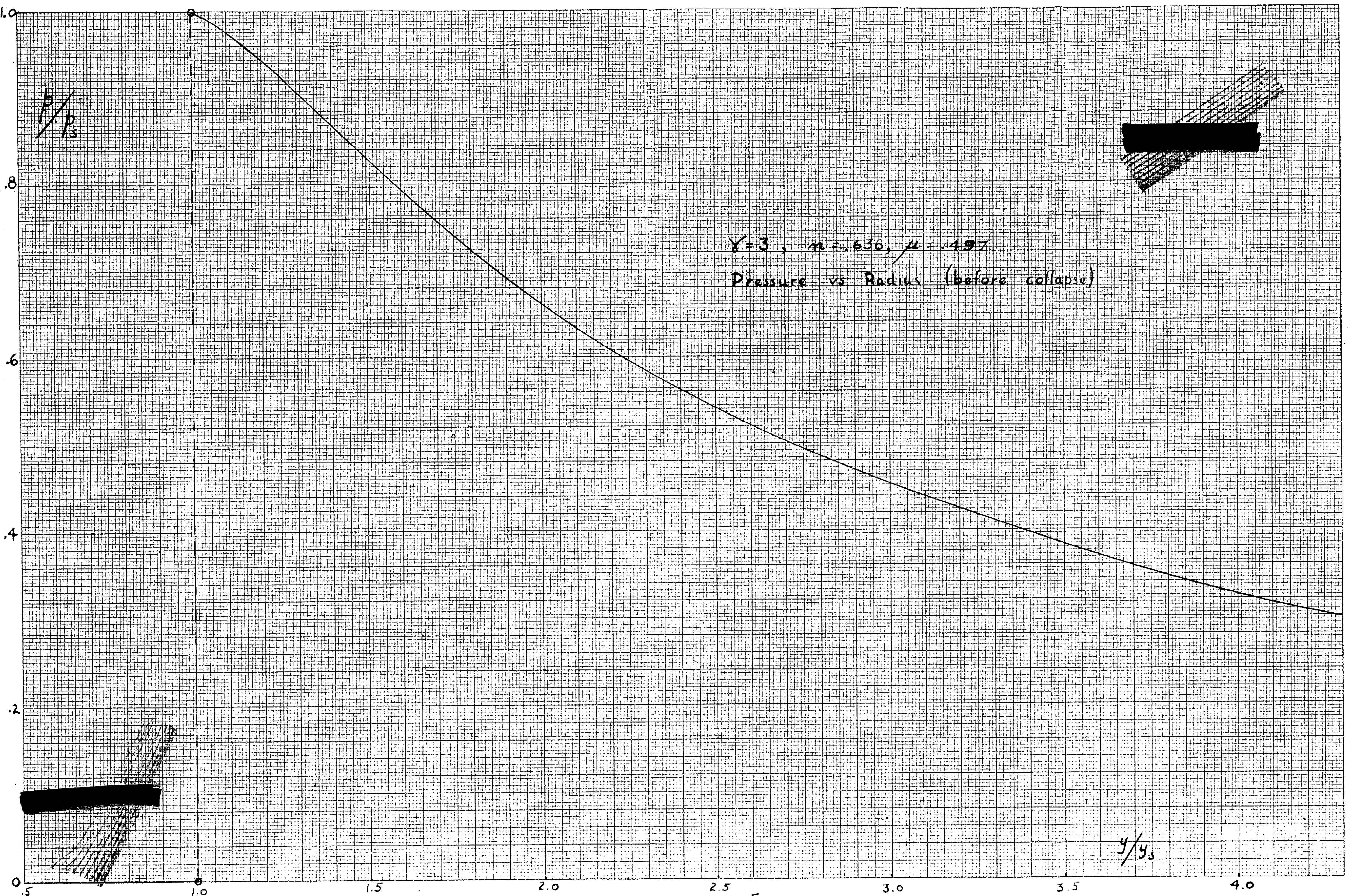

Fig. 16

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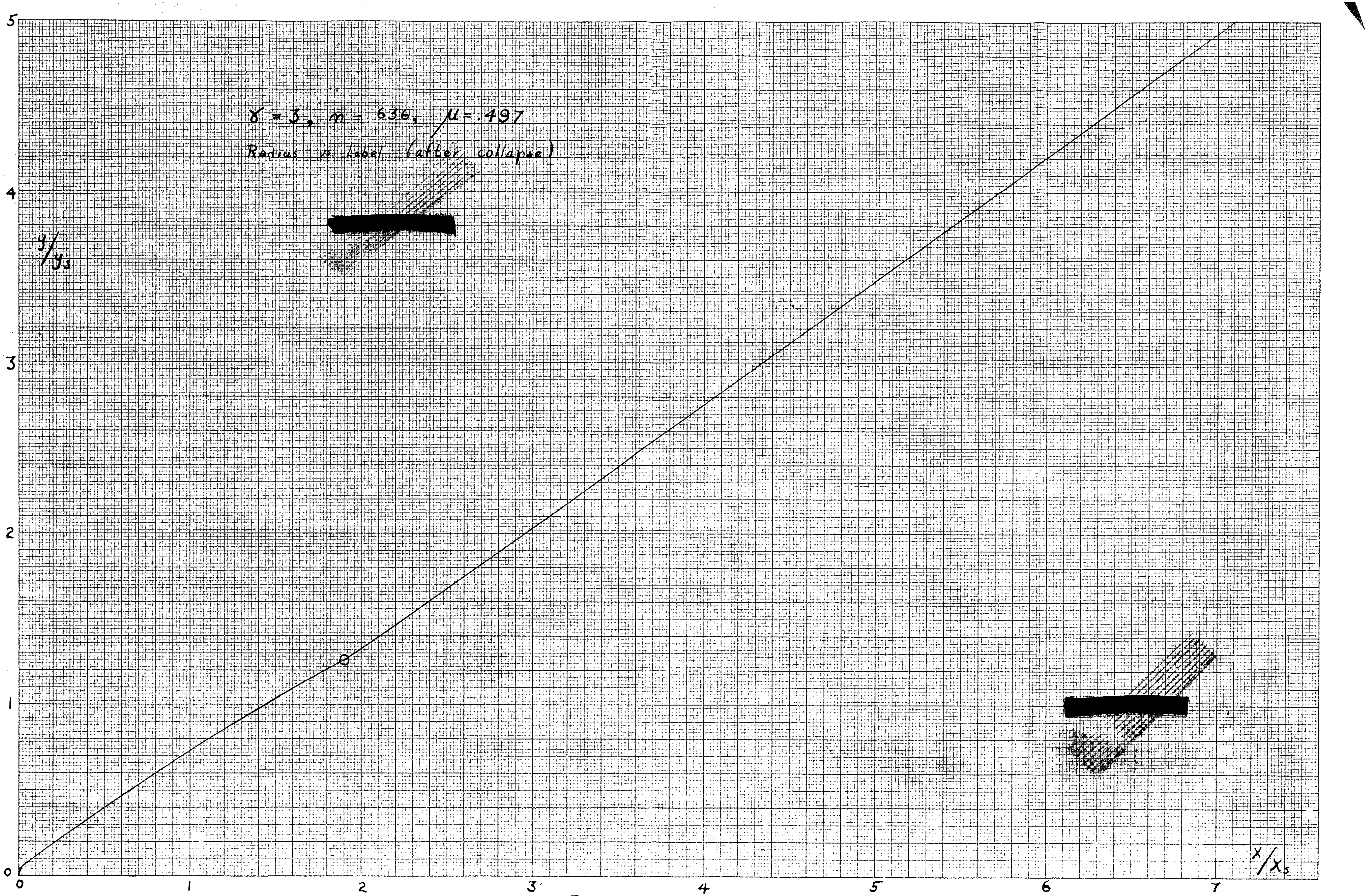

Fig. 17

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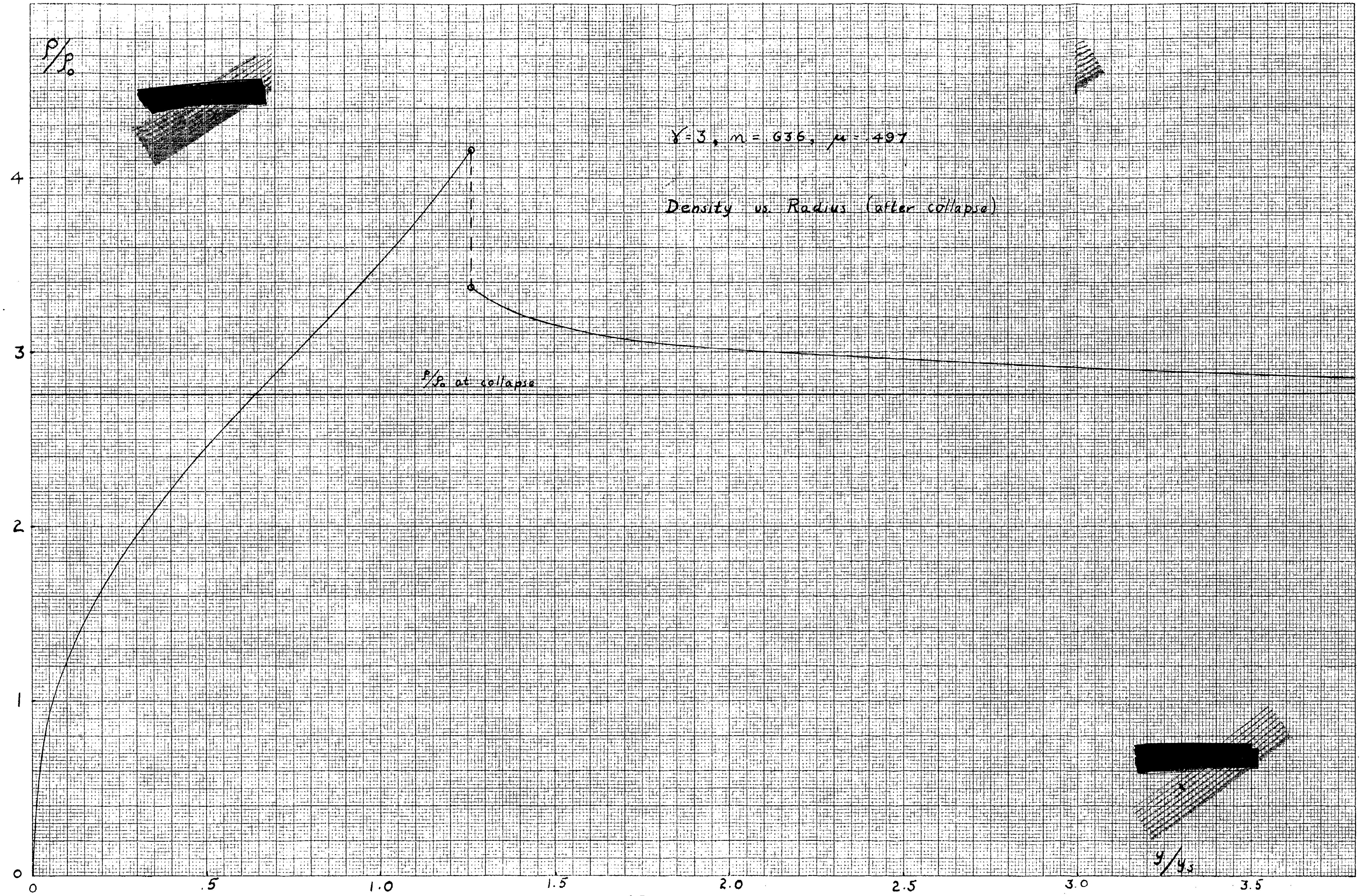

Fig. 18

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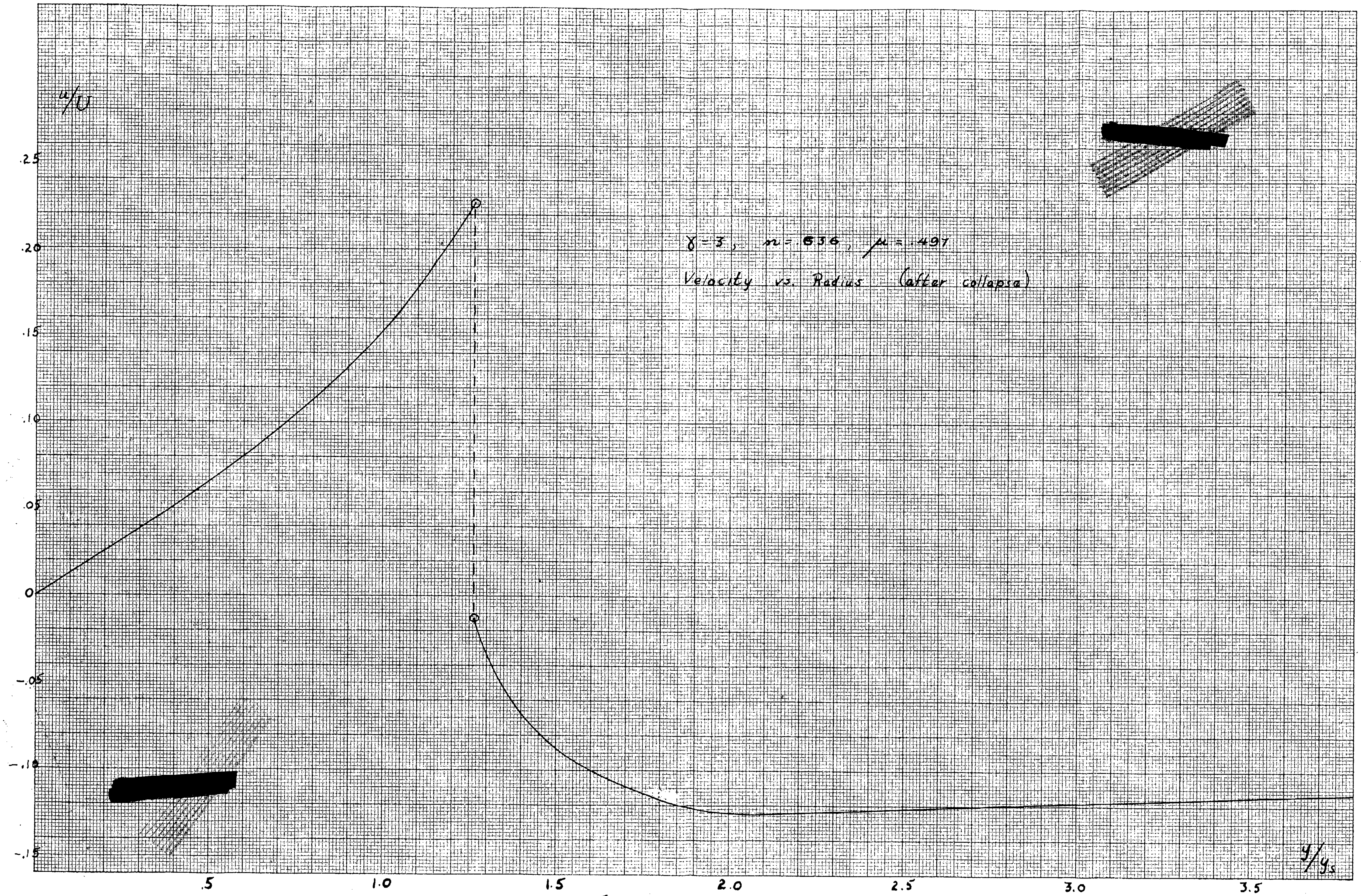

Fig 19

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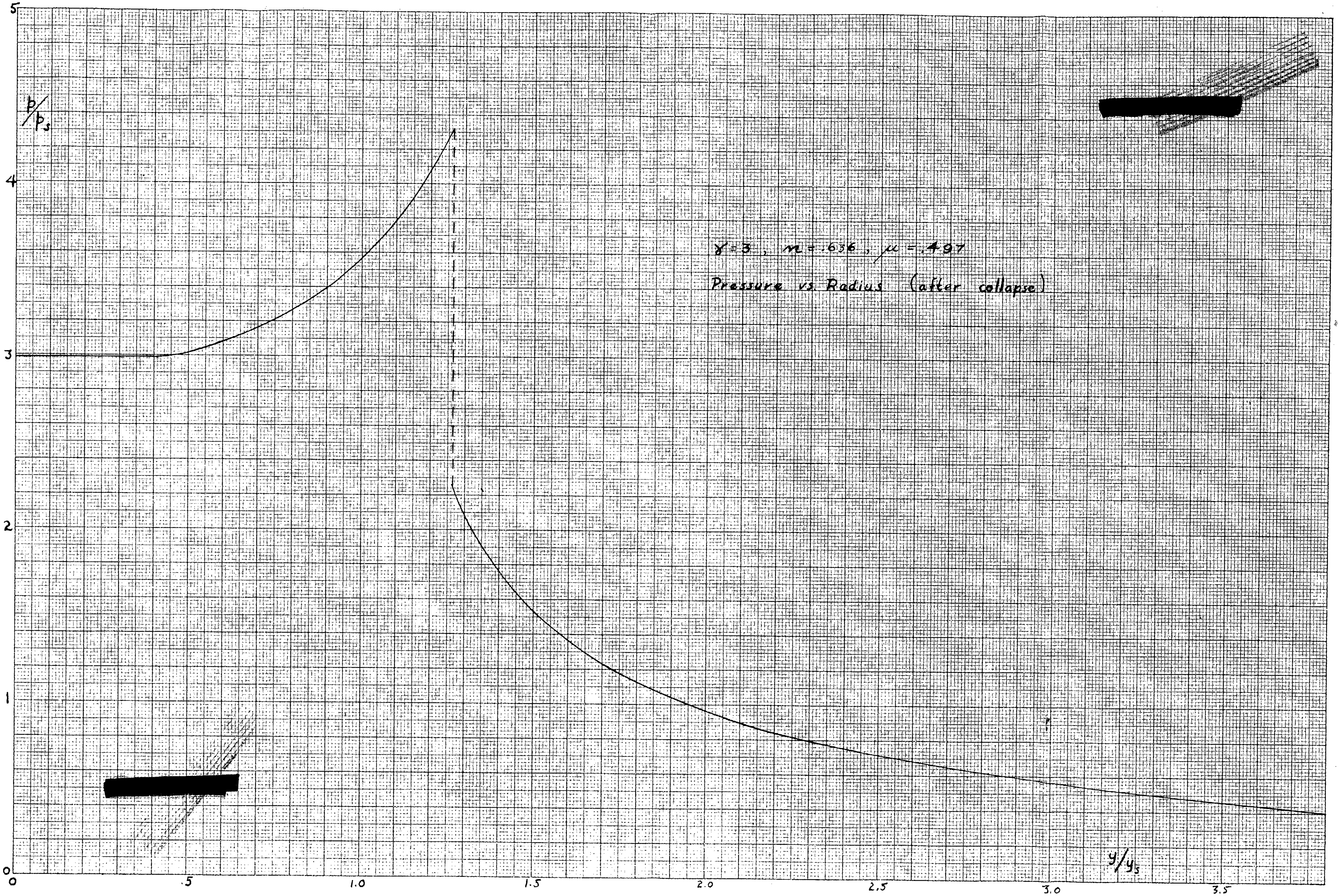

Fig. 20

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